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ELEMENTARY MATHEMATICS:

EMBRACING

ARITHMETIC, GEOMETRY,

AND

ALGEBRA,

FOR

STUDENTS IN SCIENCE CLASSES IN CONNECTION WITH THE SCIENCE AND ART DEPARTMENT.

BY

LEWIS SERGEANT, B.A., (CAMB.)

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PREFACE.

THIS book is intended as a course of Elementary Pure Mathematics, for the special use of those preparing for the Government Science and Art Examinations. It is not intended for very young boys, inasmuch as a certain amount of preliminary knowledge is expected from the student; and the explanations and examples might be held to be too difficult for those who had not at least laid a foundation of arithmetical science. But for such as usually submit themselves to the test of the "elementary stage," it is hoped that this volume will prove a serviceable and sufficient text-book. The compiler has endeavoured to enunciate sound principles, neither more nor less than are necessary for the purpose, and to illustrate these principles in the clearest manner. In particular, he has tried to render the work acceptable as a teacher's handbook, being convinced that oral class teaching is infinitely more valuable, as it is certainly becoming more and more customary, than a painful process of book-poring on the part of the student.

L. S.

LONDON, *December*, *1872*.

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MATHEMATICS.

ARITHMETIC.

Numeration and Notation.

1. If we set down one figure, or several figures side by side, we call that which we have set down a **number.** Thus 3,472,569 is a number, and represents 'three million, four hundred and seventy-two thousand, five hundred and sixty-nine.' The process of representing the number by words is called **Numeration.** The process of representing the number by figures is called **Notation.**

2. We commonly employ nine figures (in addition to 0 for 'nought' or 'zero'), because in our decimal scale of notation we have other means of representing ten and all higher numbers. The number above given will serve as an example. The 9, in the right hand place, represents nine *units,* or *ones;* the 6, in the second place from the right, represents 6 *tens;* the 5, five *hundreds;* the 2, two *thousands;* the 7, seven *tens of thousands;* the 4, four *hundreds of thousands;* the 3, in the seventh place from the right, three *millions;* and so on, until a figure in the thirteenth place would represent so many *millions of millions,* or *billions.* If, now, to the number above given we add 1, or one *unit,* we shall have altogether ten *units,* that is, one *ten.* This ten must go in the tens' place; and the number would end with 570, 'five hundred and seventy,' instead of 569. Similarly, if we had added to the same number one million, five thousand, two hundred and one—that is, 1,005, 201, we must write the result 4,477,770—that is, 'four million,

four hundred and seventy-seven thousand, seven hundred and seventy.'

3. It is clear then that the value of any figure in a number depends on its distance from the right-hand figure, *increasing* tenfold for every move to the *left*, or *decreasing* tenfold for every move to the *right*. Suppose, now, that we cease to mark off our numbers by commas after every third figure, as above, and write them thus, 3472·569. The dot between the 2 and the 5 is employed to show that the figure 2 occupies the *units'* place. Consequently the 7 represents seven *tens;* the 4, four *hundreds;* the 3, three *thousands.* And proceeding from the dot to the right, the 5 represents ten times less than if it were in the *units'* place; the 6, a hundred times less; the 9, a thousand times less. In other words, the 5 represents five *tenth-parts* of a unit; the 6, six *hundredth-parts* of a unit; the 9, nine *thousandth-parts* of a unit. Also, since the *tenth* is ten times as much as the hundredth, and the *hundredth* is ten times as much as the thousandth, the three figures on the right of the dot represent 'five hundred and sixty-nine *thousandths*' of a unit. And similarly, if the dot be moved one place to the *left*, the value of every figure will be made ten times *less;* and if the dot be moved one place to the *right*, the value of every figure will be made ten times *greater.*

4. This is what is meant by the 'decimal scale of notation,' or the 'decimal system.' The figures on the left of the dot represent a 'whole number,' and the figures on the right of the dot represent a 'fraction' of *one unit*, or (when written in this form) a 'decimal fraction.' If no dot is present, the figures will represent a whole number, and the dot might be understood to be present on the right of the units' place.

5.—Examples. Numeration and Notation.

I. Write out in words the following numbers: 17; 635; 8972; 13461; 333570; 1912568; 3015670;

2342601; 101010101; 325000010; 8570601101; 320560707; 32563010011672009.

II. Write in words: 3·1; 12·31; 625·625; 56718·01; 312·5607; 1·0001; 523467·101; 10000·00001; ·3; ·301.

III. Make the number 325·607 ten times less; ten times greater; a hundred times less; a hundred times greater;—writing out the results in words.

IV. Multiply 567·25 by 100; and divide the same number by 10000;—writing out the results in words.

V. Why are the final noughts in 6·100 unnecessary?

VI. Write in figures the following numbers: seven hundred and seventeen; eleven hundred and ninety-one; three thousand and three; thirty-one thousand and thirty-one; one hundred and one thousand and one; one million one hundred thousand one hundred and one; one billion one million one thousand one hundred and one.

VII. Write in figures: one and one tenth; fifteen and seven tenths; thirty-five and thirty-five hundredths; sixteen thousand and sixteen thousandths; one hundred and ten and fifty thousandths.

The Fundamental Rules, Applied to Whole Numbers and Decimal Fractions.

6. Rule.—*To add together two or more numbers,* set them down in a column, units under units, tens under tens, and so on toward the *left;* and if there be any decimal fractions, tenths under tenths, and so on to the *right.* Then, beginning at the right, add up the first column of figures. If the sum is less than 10, set the figure down underneath, and go on to the next column. If the sum is 10 or more, set down the right-hand figure, and carry the other to the next column. Now add up the second column, and proceed in like manner, up to the last figure-column on the left.

Thus—

378	3263·69
6391	85·001
852883	94576·32525
159	1·1
90000	12·01
949811	97938·12625
1 132	211 1

7. Rule.—*To subtract one number from another*, set the least under the greatest—units under units, &c., as above. Subtract the right-hand figure of the bottom number from the figure above it, and set down the remainder underneath. If, however, the lower figure is greater than the one above it, increase the latter by 10; then subtract, set down the difference, and decrease the top figure of the next column by 1. Proceed as before to the last figure-column on the left.

Thus—

5681253	3265·298
835185	98·5493
4846068	3166·7487
1 1 11	111 11

Here in the first example, since we cannot take 5 from 3, we borrow *ten* from the *fifty*, and take 5 from 13: the remainder being 8. Then we have to take 8 (tens) from 4 (tens), instead of from 5 (tens), because we have previously transferred one ten to the units' column.—In the second example we begin by taking 3 from 10: for the fraction ·298 means the same thing as ·2980.

8. Rule.—*To multiply one number by another.* If the multiplier be a single figure, multiply by it the units' figure of the number to be multiplied. If the product thus obtained be greater than 9, set down the units in the units' place, and carry the tens. Then multiply the tens' figure of the multiplicand by the multiplier, and add to the product the tens carried from the units' place; and so on. If there be a tens' figure in the multiplier, proceed with it in the same manner, but set down the first

figure of the result in the tens' place. If there be a hundreds' figure in the multiplier, set down the first figure of the product in the hundreds' place. And so forth. Then add up the separate results.

Thus—*(multiplicand)*	345926	342·25
(multiplier)	49	3 84
	3113334	1369 00
	1383704	27380 0
	16950374	102675
		131424

In the second example, since there are no hundredths nor tenths in the result, we dispense with the noughts; and as the multiplication of a fraction by a whole number has produced a whole number, we need not place the 'decimal point' after the units' figure 4.

9. Rule.—*To divide one number by another.* Take as many figures from the dividend, beginning from the left, as are sufficient to make a number greater than the divisor; divide this number by the divisor, and set down the quotient as the first figure of the result. If this first division leaves any remainder, place the next figure of the dividend on the right hand of this remainder, and divide the number thus made up by the divisor, setting down the quotient on the right hand of the former quotient. And so forth.

Thus—*divisor dividend*   *divisor dividend quotient*

```
        11)6934237              11)6934237(630385
quotient   630385—2 rem.           66
           30495                    33
                                    33
                                     042
                                      33
                                       93
                                       88
                                        57
                                        55
                                         2 remainder.
```

The first of these modes of arrangement is the shortest, and may be employed whenever the divisor is so small as to permit the successive acts of *division* and *subtraction* to be performed in the head. The student will do well to mark the identity of the process in the two cases.

10. When we break up the divisor into fractions, it often happens that several, or all of the successive steps give us a remainder; and the true remainder of the complete division must be found by combining the partial remainders in the manner indicated below:—

$$1260\left\{\begin{array}{l} 4)\underline{69284325} \\ 5)\underline{17321081}—1 \\ \quad 9)\underline{3464216}—1\times 4 \\ \qquad 7)\underline{384912}—8\times 5\times 4 \\ \qquad\quad 54987—3\times 9\times 5\times 4 \end{array}\right. \quad \begin{array}{l} \text{T. R.} = 1+4+160+540 \\ \qquad\ \ = 705 \end{array}$$

For the second dividend is a number of *fours;* the third, a number of *twenties;* the fourth, a number of *one hundred and eighties.* The accuracy of the result may be tested by working the example by long division, when the true remainder is the last result of the process.

The accuracy of our work in any of the fundamental processes may be tested by employing the results in the reverse process; as in the following simple example.

$$\begin{array}{ll} \textit{Division.}\ 1260)69284325(54987 & \text{Proved by}\quad 54987 \\ \qquad\qquad\quad \underline{6300} \qquad\ \textit{quot.} & \textit{Multiplication.}\ \ \underline{1260} \\ \qquad\qquad\qquad 6284 & \qquad\qquad 3299220 \\ \qquad\qquad\qquad \underline{5040} & \qquad\quad 109974 \\ \qquad\qquad\qquad 12443 & \qquad\quad 54987 \\ \qquad\qquad\qquad \underline{11340} & \qquad\qquad\qquad \underline{705}\ \text{rem.} \\ \qquad\qquad\qquad\ 11032 & \qquad\quad 69284325 \\ \qquad\qquad\qquad\ \underline{10080} & \\ \qquad\qquad\qquad\quad 9525 & \\ \qquad\qquad\qquad\quad \underline{8820} & \\ \qquad\qquad\qquad\quad\ 705\ \text{T. R.} & \end{array}$$

11. Rule.—*To multiply or divide one decimal fraction or mixed number by another.* Proceed as in the multiplication or division of integers. If after dividing there be a remainder over, or if the figures in the dividend represent in the first instance a smaller number than that represented by the figures of the divisor, assume any number of noughts after the right hand figure of the dividend, and proceed with the division until there is no remainder, or until a given number of decimal places is attained in the quotient. Then (1.) *in the result of multiplication*, point off as many decimal places as will amount to the *sum* of the decimal places in the multiplicand and the multiplier; and (2.) *in the result of division*, point off as many decimal places as will amount to the *difference* of the decimal places in the dividend and the divisor, remembering to count the noughts assumed in the dividend as representing decimal places. Thus—

```
  2·3257 multiplicand.              dividend.
    ·563 multiplier.   divisor ·25)1·32575(5·303 quotient.
   69771                           1 25
  139542                              75
 116285                               75
 1·3093691 product.                    075
                                        75
                                         ·
```

```
                      ·6125)5·2500(8·5714
                            4 9000
      (5·25 ÷ ·6125)         35000
                             30625
                              43750
                              42875
                                8750
                                6125
                                26250
                                24500
                                 1750
```

In the last example it will be observed that, though there is a remainder over, we point the quotient according to the rule, reckoning eight places of decimals in the dividend. We have really divided 5·25000000 by ·6125; our quotient is 8·5714, and our remainder is ·00001750.

12.—EXAMPLES. FUNDAMENTAL RULES.

	A	B	C
1	12	81689543	269612·5412
2	69	69991258	184383·184
3	325	28983492	126425·92
4	547	26894326	94398·64324
5	853	25785432	93639·24
6	6918	13729854	93295·4368
7	7854	11639432	998·135464
8	9237	8698751	884·53201
9	86543	1437629	800·01001
10	92785	983595	543·2548
11	8329857	480989	483·248
12	9505876	165438	48·20060

(1.) Add together the numbers in these columns, taking them two, three, four, &c., up to twelve at a time.

(2.) Subtract any number in (B) or (C) from any number above it.

(3.) Multiply any number in either column by any other number.

(4.) Divide any number in (B) by any of the first ten numbers in (A).

(5.) Divide any number in (C) by any number in (A).

(6.) Divide any number in (C) by any other number in the same column.

(7.) Prove the results in the first six cases of each exercise by the processes indicated in § 10.

Fractions Considered as Ratios.

VULGAR FRACTIONS.

13. A **ratio** is a comparison of the magnitudes of two

quantities. The numerical value of a ratio is obtained by dividing one quantity by the other. Hence every ratio represents a *quotient*, which may be obtained in the form of a whole number or a decimal fraction by the actual process of division.

Examples of ratios. The ratio of 4 to 2 is expressed as $\frac{4}{2}$; the numerical value of which is 4 *divided by* 2, $(4 \div 2) = 2$. So the ratio of 121 to 4 is expressed by $\frac{121}{4}, = 30\cdot25$. The ratio of 6s. 9d. to £1. 0s. $1\frac{1}{2}$d. is expressed by

$$\frac{6s.\ 9d.}{£1.\ 0s.\ 1\frac{1}{2}d.}, = \frac{162 \text{ halfpence}}{483 \text{ halfpence}} = \cdot3354\ldots$$

14. The ratio $\frac{1}{2}$ signifies $1 \div 2$, or "one unit divided into two equal parts." Thus $\frac{1}{2}$ may be called "one-half;" and therefore represents a *fraction of one unit.* So $\frac{2}{3}$ signifies $2 \div 3$, or "two units divided into three equal parts," which is equivalent to "twice as much as one-third of one unit," or "two-thirds." Similarly $\frac{15}{17}$ signifies "fifteen times the seventeenth part of one unit." All ratios represented in this form are called **vulgar** or **ratio fractions.** They are called **proper** vulgar fractions when their value does not exceed one unit; as $\frac{1}{4}$, $\frac{13}{14}$;—and **improper** vulgar fractions when their value exceeds one unit; as $\frac{5}{4}$, $\frac{15}{14}$.

15. The upper number in a vulgar fraction is called the **numerator,** because it gives us the *number* of fractional parts taken; the lower number is called the **denominator,** because it gives us the *name* of the fraction: that is, it shows into how many equal parts the unit is considered to be divided. Comparing decimal fractions with vulgar, we may conceive a decimal to have for a denominator 1, followed by as many 0's as there are figures on the right hand of the dot, whilst these latter "significant" figures may be regarded as the numerator.

$$\text{Thus } \cdot1 = \tfrac{1}{10};\ \cdot25 = \tfrac{25}{100} = \tfrac{1}{4};\ 1\cdot3 = \tfrac{13}{10}.$$

16. If we wish to convert a vulgar fraction into a

decimal, we have to find a new numerator and denominator bearing to each other as nearly as possible the same ratio as the old ones, at the same time that the *denominator* is some power of 10. Occasionally this may be done by inspection. Take, for instance, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$. Here the equivalent ratios will manifestly be $\frac{5}{10}$, $\frac{2}{10}$, $\frac{4}{100}$, $\frac{8}{1000}$ respectively; and the decimals will be expressed as ·5, ·2, ·04, ·008. But, if the decimal ratio is not immediately manifest, we must divide the numerator of the vulgar fraction by the denominator, supplying 0's in the dividend successively as long as we require them. Take the case of $\frac{3}{7}$. When we have supplied one 0 in the dividend, the decimal is ·4 ($=\frac{4}{10}$). When we have supplied two 0's, the decimal is ·42 ($=\frac{42}{100}$). When we have supplied three 0's, the decimal is ·428 ($=\frac{428}{1000}$); and so on. It is manifest that each successive figure in the decimal makes the latter more nearly equal to the value of $\frac{3}{7}$: the respective errors in the above cases being $\frac{1}{35}$, $\frac{3}{350}$, $\frac{1}{1750}$.

$$\begin{array}{r@{}l}
7)3 & \;(\cdot 4285\ldots \\
2\,8 & \\
\hline
2\,0 & \\
1\,4 & \\
\hline
6\,0 & \\
5\,6 & \\
\hline
4\,0\ldots &
\end{array}$$

If the division terminates without a remainder, we have found a decimal which is the exact equivalent of the vulgar fraction. If we have reason to believe that the division will not terminate, we may stop short at any point, knowing that after four or five places of decimals the difference in the value of the vulgar and decimal fractions will be extremely small.*

Whenever we stop the division, we must point the decimal according to the rule given in § 11, accounting for the 0's assumed in the dividend.

When the division does not terminate without remainder, the decimal is called **indeterminable.** As will hereafter be shown, the decimal will be indeterminable whenever the denominator of the vulgar fraction

* A further explanation of this process is given in the Appendix to the *Algebra.*

contains any other factor than 2 and 5, the measures of 10 (see § 28).

17. Hence we have the following rules. A decimal fraction may be converted into a vulgar fraction by taking the significant figures of the decimal for a numerator, and for a denominator 1, followed by as many 0's as there are decimal places on the right of the point.

A vulgar fraction may be converted into a decimal by dividing the numerator by the denominator, supplying 0's in the numerator as often as required, and pointing the result at any assigned step according to the number of 0's then assumed.

18. The ratio, or comparison of magnitude, between the numerator and denominator of a vulgar fraction, must be equivalent in value to the ratio between the doubles, trebles, quadruples, &c., of these quantities;* since the one will still be the same number of times greater or less than the other.

Hence, a vulgar fraction may be reduced to lower terms, without altering its value, by dividing numerator and denominator by any common divisor.

Thus $\frac{25}{100} = \frac{1}{4}$; $\frac{225}{100} = \frac{200}{100} + \frac{25}{100} = 2\frac{1}{4}$.

This reduction has frequently to be made after converting a decimal into a vulgar fraction.

19. A whole number followed by a fraction is called a **mixed number**. Thus $2{\cdot}25$, or $2\frac{1}{4}$, is a mixed number. It is clear that a mixed number is equivalent to an improper fraction.

Hence, an improper fraction may be reduced to a mixed number by dividing the numerator by the denominator. The integral quotient will be the integral part of the mixed number, and the remainder left from the numerator will be the numerator of the new fraction, which will have the original denominator.

(Thus $\frac{629}{12} = 52\frac{5}{12}$.)

* See *Geometry*, Ax. 2.

20. Again, since one unit is equivalent to two halves, three thirds, four fourths, &c., two units are equivalent to four halves, six thirds, &c.; and so forth.

Thus, $2 = \frac{4}{2}$; therefore $2\frac{1}{2} = \frac{5}{2}$; $12\frac{5}{7}\ (= \frac{12 \times 7 + 5}{7}) = \frac{89}{7}$.

21. *Hence*, a mixed number may be reduced to an improper fraction, by multiplying the whole number by the denominator, adding in the numerator, and setting down this sum as a new numerator over the original denominator.

22. *Hence, also*, a whole number may be expressed as a fraction with any given denominator.

Thus, $5 = \frac{5 \times 7}{7} = \frac{35}{7}$; or, according to the last rule,

$$5 = 5\tfrac{0}{7} = \tfrac{35}{7}.$$

Multiplication and Division of Ratio Fractions.

23. The numerical value of the ratio of 3 to 4, or of $\frac{3}{4}$, will be increased by increasing the numerator, and decreased by decreasing the numerator. Again, it will be increased by decreasing the denominator, and decreased by increasing the denominator. Thus—

$$\tfrac{3}{4} \times 5 = \tfrac{15}{4}, \text{ or, } = \frac{3}{4 \div 5}\text{; and } \tfrac{3}{4} \div 5 = \frac{3 \div 5}{4}, \text{ or } = \tfrac{3}{20}.$$

24. *Hence*, to multiply a vulgar fraction by a whole number, *multiply the numerator*, or *divide the denominator*.—To divide a vulgar fraction by a whole number, *divide the numerator*, or *multiply the denominator*.

(In either case it is better to *divide* if possible, since this gives the result in its lowest terms. Thus—

$\frac{9}{20} \times 5 = \frac{9}{4} = 2\frac{1}{4}$; $\frac{19}{20} \div 2 = \frac{19}{40}$, where the division cannot be performed.)

25. Again, it is clear that $\frac{2}{3} \times \frac{5}{6} = \frac{2 \times 5}{3} \times \frac{1}{6} = \frac{2 \times 5}{3 \times 6} = \frac{10}{18} = \frac{5}{9}$; $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \div 5 \times 6 = \frac{2}{3} \times \frac{6}{5}$.

26. *Hence*, to multiply one vulgar fraction by another, multiply the numerators for a new numerator, and the denominators for a new denominator.—To divide one

vulgar fraction by another, invert the divisor-fraction, and multiply.

27. It was just seen that $\frac{2}{3} \times \frac{5}{6} = \frac{10}{18} = \frac{5}{9}$. Now the factor 2, which was struck out of 10 and 18, might have been struck out of the original fractions $\frac{2}{3}$ and $\frac{5}{6}$. Thus—

$$\frac{\cancel{2}}{3} \times \frac{5}{\underset{3}{\cancel{6}}} = \frac{5}{9}. \quad \text{So, } \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{5}}{\underset{3}{\cancel{6}}} \times \frac{\cancel{9}}{\cancel{25}} \times \frac{\cancel{5}}{\cancel{3}} = \frac{1}{3}.$$

This process is called *cancelling* the factors common to numerators and denominators.

28. A **factor** of a number is one which will divide it without remainder. Thus 2, 3, 4, and 6 are separately factors of 12; whilst 2 and 6, or 3 and 4, are factors composing, or equivalent to, 12. The factors of a number are also called its **measures.**

29. Again, 2 is called a **common measure** of 12, 20, and 32; and 4 is another common measure of the same three numbers. It is clear that 4 is the **greatest common measure** of these three numbers.

30. A number which is divisible by another without remainder is called a **multiple** of it. Thus 12, 20, and 32 are multiples of 2.

31. Again, 48 is a **common multiple** of 6, 8, and 12. So also is 24. And it is clear by inspection that 24 is the **least common multiple** of 6, 8, and 12.

32. A number which has no measures is called a **prime** number. Thus 3, 7, 19 are primes.

Addition and Subtraction of Ratio Fractions.

33. In order to express the sum of two or more vulgar or ratio fractions as a single fraction it is clearly necessary that the fractions should first be reduced, so as to have one common denominator. (Thus we cannot immediately add together $\frac{1}{7}$ and $\frac{1}{4}$. It is necessary to observe that $\frac{1}{7} = \frac{4}{28}$, and that $\frac{1}{4} = \frac{7}{28}$. Then $\frac{1}{7} + \frac{1}{4} = \frac{4+7}{28} = \frac{11}{28}$.)

34. *Hence,* in order to add together two or more vulgar

fractions, find the *least* common multiple of their denominators, and take this for the common denominator. Multiply each numerator by the factor introduced into the corresponding denominator; and add together the numerators thus obtained.—Proceed in a similar manner for subtraction. (Thus—

$$\tfrac{1}{2} + \tfrac{3}{4} + \tfrac{5}{6} + \tfrac{2}{3} + \tfrac{7}{8} = \tfrac{12+18+20+16+21}{24} = \tfrac{87}{24} = 3\tfrac{15}{24}$$
$$= 3\tfrac{5}{8};\ \tfrac{3}{4} - \tfrac{5}{8} = \tfrac{6-5}{8} = \tfrac{1}{8}.)$$

35. If the denominators are large, or many in number, so that the least common multiple is not evident by inspection, it must be found by another method, which generally requires the use of the greatest common measure.

To find the G. C. M. of two numbers, divide the greatest by the least, then the divisor by the remainder, and so forth, until there is no remainder. The last *divisor* will be the G. C. M.*—

If the G. C. M. of more than two numbers be required, find the G. C. M. of any two; then the G. C. M. of the result and the third; then the G. C. M. of the last result and the fourth; and so on. The last G. C. M. will be the G. C. M. of all.

Thus—G. C. M. of 1107 and 1845.

```
1107)1845(1
     1107
      738)1107(1
           738
           369)738(2
               738
```

G. C. M. = 369.

36. To find the L. C. M. of two numbers, find their G. C. M.; divide one number by the G. C. M., and multiply the quotient by the other number. (Thus, to find the L. C. M. of 1107 and 1845. Their G. C. M. is 369. Therefore the L. C. M. $= \frac{1107}{369} \times 1845 = 3 \times 1845 = 5535$.)—For more than two numbers, proceed as for G. C. M., two at a time.

37. All that is necessary is to provide that no common factor of any two of the original numbers enters more

* The reason for this Rule is best explained by Algebraical symbols. See *Algebra*, Appendix.

than once into the composition of the L. C. M. Hence the following method is often simpler in practice :—

Required the L. C. M. of 36, 54, 50, 9, 120, 252.

$$\left.\begin{array}{r}36 = 2 \times 2 \times 3 \times 3 \\ 54 = 2 \times 3 \times 3 \times 3 \\ 50 = 2 \times 5 \times 5 \quad\quad\quad \\ 9 \quad\quad\quad\quad\quad\quad\quad \\ 120 = 2 \times 2 \times 2 \times 3 \times 5 \\ 252 = 2 \times 2 \times 3 \times 3 \times 7\end{array}\right\} \begin{array}{l}\text{L.C.M.} = (2 \times 2 \times 2) \times (3 \times 3 \times 3) \\ \quad\quad\quad\quad \times (5 \times 5) \times 7. \\ \quad\quad = 8 \times 27 \times 25 \times 7. \\ \quad\quad = 37800.\end{array}$$

Here the numbers are all broken up into primes. We take the $2 \times 2 \times 2$ from 120; the $3 \times 3 \times 3$ from 54; the 5×5 from 50; the 7 from 252; omitting these factors when they occur again. The 9 is cancelled at first, because it is contained in 36.

38. A complex fraction is one in which the numerator or denominator is itself a fraction; as in the following examples:—

$$\frac{\frac{2}{3} \times \frac{5}{7}}{1\frac{9}{21}} = \frac{2}{3} \times \frac{5}{7} \div 1\frac{9}{21} = \frac{2}{3} \times \frac{5}{7} \times \frac{21}{30} = \frac{1}{3}.$$

$$\frac{2(\frac{3}{5} \text{ of } \frac{6}{11})}{\frac{1}{77} \text{ of } \cdot\dot{8}} = \frac{2}{1} \times \frac{3}{5} \times \frac{6}{11} \div (\frac{1}{77} \times \frac{8}{9}) = \frac{2}{7} \times \frac{3}{5} \times \frac{6}{11} = \frac{77}{1} \times \frac{9}{8}$$

$$= \frac{567}{10} = 56\frac{7}{10}.$$

39. The word "of" after a fraction is a sign of multiplication. Thus, $\frac{3}{7}$ of 21 = three sevenths of 21 = $3(21 \div 7)$ $= 3 \times 3 = 9$.

40.—Examples on Fractions.

I. (§ 15). Reduce to decimal fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$; $\frac{7}{8}$, $\frac{9}{25}$, $\frac{11}{125}$, $\frac{101}{625}$, $\frac{15}{16}$, $\frac{121}{3125}$.

II. (§ 16.) Reduce to vulgar fractions ·625, ·815, ·362, ·8, ·125, ·3125; 63·25, 8·45, 10·01, 5·0001.

III. (§ 17.) Reduce to lowest terms $\frac{9}{12}$, $\frac{8}{24}$, $\frac{16}{64}$, $\frac{15}{75}$, $\frac{36}{144}$; $\frac{126}{154}$, $\frac{12321}{555000}$, $\frac{132}{1728}$, $\frac{84375}{91500}$.

IV. (§ 19.) Reduce to improper fractions $2\frac{1}{6}$, $3\frac{2}{3}$, $4\frac{1}{7}$, $5\frac{1}{11}$, $6\frac{1}{8}$; $23\frac{6}{13}$, $45\frac{5}{8}$, $9\frac{322}{610}$, $8\frac{123}{148}$, $82\frac{31}{124}$.

V. (§ 21.) Reduce to mixed numbers $\frac{7}{4}$, $\frac{8}{3}$, $\frac{21}{5}$, $\frac{82}{11}$, $\frac{96}{23}$, $\frac{137}{25}$, $\frac{695}{15}$, $\frac{834}{16}$, $\frac{9125}{625}$.

VI. (§ 22.) Express 7 with denominators 6, 3, 12, 21; 12 with denominators 9, 13, 20; 32 with denominators 3, 4, 11, 60.

VII. (§ 23–27.) $\frac{3}{7} \times 5, 7, 12$; $\frac{12}{13} \times 8, 12, 26, 169$; $2\frac{1}{3} \times 6, 8, 11$; $6\frac{5}{12} \times 8, 16, 24$;—$\frac{2}{3} \times \frac{7}{8} \times \frac{6}{15} \times \frac{40}{42}$; $\frac{5}{9} \times \frac{6}{11} \times \frac{11}{100}$; $3\frac{1}{3} \times 5\frac{1}{10} \times \frac{12}{17} \times 3\frac{1}{3}$;—$\frac{1}{2} \div 7, 8, 12$; $\frac{7}{12} \div 8, 14, 21$; $\frac{4}{11} \times \frac{8}{9} \div \frac{6}{11} \times \frac{8}{13} \times 1\frac{2}{9} \div 832$.

VIII. (§ 29–32.) The G. C. M. of 24, 96, 144; of 22, 121, 110; of 81, 729, 336; of 147, 700, 497, 350; and the L. C. M. of 6, 8, 12; of 5, 16, 10; of 9, 12, 120; of 8, 9, 12, 72, 288.

IX. (§ 35.) G. C. M. of 1130 and 791; of 72 and 1728; of 12321 and 999; of 231 and 3465; of 3915 and 4437.

X. (§ 36, 37.) L. C. M. of 24 and 48; of 96 and 144; of 12, 16, 24; of 18, 22, 40, 99; of 8, 12, 16, 18, 20, 24; of 6, 7, 8, 9, 10, 12, 560, 1080.

XI. $\frac{1}{3} + \frac{7}{8} + \frac{5}{6} + \frac{3}{16}$; $\frac{7}{9} + \frac{10}{11} + \frac{3}{5}$; $\frac{3}{8} + \frac{5}{12} + \frac{1}{16} + \frac{3}{20}$; $2\frac{1}{3} + 4\frac{1}{6} + 8\frac{2}{9} + 7\frac{2}{15}$; $\frac{6}{10} + \frac{3}{4} + 7\frac{1}{8} - \frac{3}{12}$; $8\frac{1}{9} + 5\frac{1}{4} - 6\frac{1}{8}$.

XII. (§ 38.) Reduce to lowest terms $3(\frac{1}{3} \text{ of } 18)$; $\frac{7}{11}$ of 42; $\dfrac{9\frac{1}{6} \times 3\frac{1}{3}}{4\frac{1}{18}}$; $\dfrac{\frac{1}{3} \text{ of } 61 \times 9}{3\frac{1}{3} \times 6\frac{1}{10}}$; $\dfrac{\frac{7}{11}[3 - \frac{1}{11}(5\frac{1}{3} - \frac{2}{3})]}{1\frac{1}{21}}$.

Proportion.

41. When the ratio between two quantities is equal in numerical value to the ratio between two other quantities, the four quantities are said to be in **proportion.** (Thus, $\frac{2}{3} = \frac{6}{9}$; and 2, 3, 6, and 9 are in proportion.) This may be expressed by saying that "2 is to 3 as 6 to 9;" or symbolically, $2 : 3 :: 6 : 9$.

42. Multiply the equal fractions $\frac{2}{3}$ and $\frac{6}{9}$ by 3×9;

the products must be equal. Therefore $\frac{2 \times 3 \times 9}{3} = \frac{6 \times 3 \times 9}{9}$; that is $2 \times 9 = 6 \times 3$. *Hence* the product of the first and last terms of a proportion (the **extremes**) is equal to the product of the second and third terms (the **means**).

43. Again, since $2 : 3 :: 6 : 9$, and therefore $2 \times 9 = 3 \times 6$, $2 = \frac{3 \times 6}{9}$; $9 = \frac{3 \times 6}{2}$; $3 = \frac{2 \times 9}{6}$; $6 = \frac{2 \times 9}{3}$. *Hence*, if the two means of a proportion are known, and one extreme, the remaining extreme may be found by dividing the product of the means by the given extreme. Similarly, if the two extremes of a proportion are known, and one mean, the remaining mean may be found by dividing the product of the extremes by the given mean.

44. It is manifest that the two means of a proportion may be interchanged without affecting its truth; as also may the two extremes. (Thus, since $2 : 3 :: 6 : 9$, so $2 : 6 :: 3 : 9$; $9 : 3 :: 6 : 2$; $9 : 6 :: 3 : 2$. Similarly, since $\frac{2}{3} = \frac{6}{9}$, so $\frac{2}{6} = \frac{3}{9}$; $\frac{9}{3} = \frac{6}{2}$; $\frac{9}{6} = \frac{3}{2}$.)

45. We have said that if (1st) : (2d) :: (3d) : x, x, $= \frac{(2d) \times (3d)}{1st}$. If, then, any ratio is given, together with a 3d term, we can find a *fourth proportional* to them; that is, we can find a 4th term bearing the same ratio to the 3d as the 2d bears to the 1st.

46. Suppose we have a question of this kind. "The wheel of a locomotive slips seven per cent. of its journey. How far does it slip in 1,000 miles?"

Here the given ratio is that of 7 to 100 (7 "*per centum*"), and the given third term is 1,000. Representing the required number as x, we are told that—

$$x \text{ is to } 1{,}000 \text{ as } 7 \text{ to } 100$$

$$\text{or,} \quad x : 1{,}000 :: 7 : 100 \ldots\ldots\ldots\ldots \text{(A)}$$

$$\therefore x \times 100 = 7 \times 1{,}000$$

$$\therefore \quad x = \frac{7000}{100} = 70.$$

47. It is convenient to set x in the 4th term, and in the third term the number corresponding in its nature to

x. Now x = the *amount slipped* out of 1,000; and 7 the *amount slipped* out of 100. Thus the 3d and 4th terms will be $:: 7 : x$. Comparing this ratio with (A), we shall conclude from § 44 that the proportion must now stand—

$$100 : 1{,}000 :: 7 : x \ldots\ldots\ldots\ldots\ldots\text{(B)}$$

It would indeed have followed from the definition of *proportion* that as x is (manifestly) to be > 7, the 2d term must be $>$ the 1st.

48. The ratio of the greater to the less is called a "ratio of greater inequality;" and the ratio of the less to the greater a "ratio of less inequality." It is clear that the second ratio of a proportion must be *of the same kind* as the first; that is, either of *greater* or of *less* inequality.

49.—Examples on Proportion.

I. Find a fourth proportional to 2, 3, 4; to 6, 9, 12; to 8, 11, 22; to 6, 9, 15; to 3, $\frac{1}{3}$, $\frac{1}{9}$; to $2\frac{1}{2}$, $6\frac{1}{3}$, 2.

II. (1.) If 7 men count 23,165 shillings in 5 minutes, how long would 5 men be counting them?

(2.) If 8 sheep eat 596 turnips in a certain time; how many would 12 eat?

(3.) If I lose 5 pence out of every 100; how many shall I lose out of 1750?

(4.) If 32 men die out of a thousand; how many die in a town of 6125 inhabitants?

(5.) If 5 out of 11 of my apples are bad, and I have 968 apples; how many are bad?

(6.) If 16 out of every 72 shillings represent profit; how much is the gain per cent.?

(7.) If I earn 12 shillings in 25 days; how many do I earn in 625 days?

(8.) If I lose 12 per cent. of my books every year; how many do I lose out of a library of 725 volumes?

Recurring Decimals.

50. It has been shown that a decimal fraction may be expressed as a ratio, and a ratio as a decimal fraction. It will be found, however, that some ratios cannot be *precisely* expressed in a decimal form.

Thus—

$$\begin{array}{llllll}
 & 3)1\cdot0(\cdot33\ldots & & 7)1\cdot0(\cdot142857\ldots & & 9)1\cdot0(\cdot1\ldots \\
 & 9 & & 7 & & 9 \\
(\tfrac{1}{3}) & \overline{10} & & \overline{30} & (\tfrac{1}{9}) & \overline{1}\ldots \\
 & 9 & & 28 & & \\
 & \overline{1} & & \overline{20} & & \\
 & & (\tfrac{1}{7}) & 14 & & \\
 & & & \overline{60} & & \\
 & & & 56 & & \\
 & & & \overline{40} & & \\
 & & & 35 & & \\
 & & & \overline{50} & & \\
 & & & 49 & & \\
 & & & \overline{1}\ldots & &
\end{array}$$

51. In each of these examples it is clear that the division would never come to an end, the same figure or figures being repeated in the quotient periodically. Such fractions are called **recurring** or **circulating** decimals; and they are represented by setting a dot over the figure which recurs, or over the first and last of the figures which recur. (Thus $\frac{1}{3} = \cdot\dot{3}$; $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$; $\frac{1}{9} = \cdot\dot{1}$.)

52. In applying the fundamental rules to these decimals, it is clear that we can only obtain accuracy to a certain number of decimal places; and the result of any example into which a recurring decimal has entered must itself (if not reduced to a ratio) be a recurring decimal.

53. The addition and subtraction of recurring decimals is simple, if two or three more places of decimals are taken than the number required in the result, so as to obtain the proper form of the resulting recurring decimal.

54. Multiplication and division are more complicated; and in practice it is always better to reduce the recurring decimals to vulgar fractions, and, after obtaining a result, to reduce it again, if necessary, to a recurring decimal.

55. RULE.—*To reduce a recurring decimal to a vulgar fraction.* Take the recurring figures for the numerator, and an equal number of *nines* for the denominator.

(Thus $\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$; $\cdot\dot{1}4285\dot{7} = \frac{142857}{999999} = \frac{1}{7}$; $\cdot\dot{1} = \frac{1}{9}$.)

56. It will sometimes be found that a decimal fraction consists of one or more figures which do not recur, followed by one or more figures which do recur.

(Thus, $\frac{707}{1650} = \cdot 42\dot{8}\dot{4}$.)

57. RULE.—*To reduce a mixed recurring decimal to a vulgar fraction.* For the numerator subtract the non-recurring figures from the whole of the figures representing the fraction; and for the denominator take as many *nines* as there are recurring figures, followed by as many *noughts* as there are non-recurring figures.*

(Thus, $\cdot 42\dot{8}\dot{4} = \frac{4284-42}{9900} = \frac{4242}{9900} = \frac{707}{1650}$.

58.—RECURRING DECIMALS.

I. Add $3\cdot\dot{1} + 14\cdot\dot{5}2\dot{5} + 625\cdot\dot{1}4572\dot{8}$; $5\cdot\dot{3} + 17\cdot\dot{6} + 3\cdot4\dot{1}23\dot{1}$; $176\cdot23\dot{4}\dot{5} + 8972\cdot61\dot{2}3\dot{4} + 71\cdot\dot{9}\dot{2}$, correct to 5 places.

II. Reduce to decimal fractions $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$; $\frac{5}{7}$, $\frac{4}{9}$, $\frac{5}{11}$, $\frac{13}{21}$; $\frac{8}{65}$.

III. Reduce to vulgar fractions $\cdot\dot{8}$, $\cdot1\dot{6}$, $\cdot3\dot{2}\dot{7}$, $16\cdot6\dot{3}4\dot{5}$, $\cdot\dot{8}3\dot{5}$, $\cdot912\dot{6}\dot{5}$.

IV. Multiply $\cdot\dot{1}4285\dot{7}$ by 7, 63, 81; $62\cdot1\dot{2}\dot{3}$ by 4, 9, 95; $\cdot\dot{1}$ by $\cdot1\dot{6}$; $\cdot\dot{8}$ by $\cdot1\dot{6}2\dot{5}$, correct to 5 places; and prove the results by reducing the decimals to ratio frac-

* The reason for these two rules is best explained with the help of algebraical symbols. See *Algebra*, Appendix.

tions, multiplying as ratio fractions, and bringing the resulting ratio to a decimal.

V. Divide $\cdot 1\dot{6}$ by 5, 6, 11; $\cdot 32\dot{4}\dot{2}$ by 8, 9, 172, correct to 5 places; and prove the results as in the last case.

Evolution.—Square Root.

59. It will be found on trial that if a number of one digit be multiplied by itself, the resulting square number will consist of *one* or *two* digits. The square of a number of two digits will consist of *three* or *four* digits. The square of a number of three digits will consist of *five* or *six* digits; and so forth. Therefore, in proceeding back from the square to the number squared, or the **square root**, we shall know beforehand that this root will contain one digit for every two in the square. And it will appear, from the principles of decimal notation, that the units' figure of the root will be derived from the units', or units' and tens' figures of the square; the tens' figure of the root from the hundreds', or hundreds' and thousands' figures of the square; and so on.

60. We may therefore begin by pointing off the square number in periods of two figures, starting from the right. The process will then be as follows:—

RULE.—Take the greatest square number out of the first period on the left. Put the square root of this number in the result; set the square number under the first period, and subtract. Bring down the next period, and set it on the right of the remainder. Consider this increased remainder as a dividend, and for a trial divisor take *twice the first figure of the result*. Try this divisor into the new dividend *as far as what would represent its tens' place*. (Do not, however, accept a quotient if its product approaches too close to the partial trial dividend. Allow for a figure which may afterwards be carried from the right.) Set this quotient down as the

The first nine square numbers.	
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81

second figure of the result, and also on the right hand of the trial divisor. Then multiply the whole divisor by the last obtained figure of the result; set the product under the dividend, and subtract. Now, for a new trial divisor double the number already represented in the result, and proceed as before.

EXAMPLE.—Find the square root of 14814801.

```
        14814801(3849
         9
  68 | 581
     | 544
 764 | 3748
     | 3056
7689 | 69201
     | 69201
```

Here it will be seen that the first trial divisor 6 is contained 9 times in 58; but as the subsequent multiplication of 9 by 9 would give 8 to be carried to the left, thus producing $6 \times 9 + 8$ ($= 62$), we must not accept the nine, but take 8 in its stead. For convenience, instead of doubling the result at each step for the new trial divisor, we may simply double the units' figure of the last divisor (the rest having been already doubled), adding in, if necessary, 1 to the tens' figure.

61. If there is no exact square root of the number given, we may approach its value more nearly by working out a certain number of decimal places. The new periods to be brought down will of course be 00; the first such period producing *tenths* in the result, the second, *hundredths;* and so on.

62. If we have to extract the square root of a decimal mixed number, the process will be exactly the same: the first period-dot going over the units' figure, and the others on alternate figures, right and left.

Cube Root.

63. The method of finding the cube root of a number is based on similar principles to those stated in § 59. In this case the periods will consist of three figures. We begin by taking the greatest cube number in the first

The first nine cube numbers.

1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729

period. For a trial divisor we take *three times the square of the first figure* in the result. Having tried this divisor into the dividend, up to what would represent its *hundreds'* place, and so found a second figure of the result, we now add to the divisor *three times the product of the first and second figures* of the result, setting this triple product one place to the right of the former one. Thirdly, we add to these *the square of the last figure* of the result, setting it down one place to the right of the former one. We now add up these three components of the divisor, and multiply the sum by the last figure of the result; and so forth.

		$\dot{4}86\dot{2}7{\cdot}12\dot{5}(36{\cdot}5$
27		27
54		21627
36		19656
3276		1971125
		1971125
$3 \times (36)^2 = 3888$		
$3 \times (5 \times 36) = 540$		
25		
394225		

64. The difficulty of discovering each new figure of the result, by means of the trial divisor, is considerably greater than in the process for square root. It is often necessary to experiment on the slate or paper, especially when there are more than three periods.*

65. The remarks made above, in reference to the extraction of decimal places in the root, are equally applicable here.

66.—Examples on Evolution.

I. Extract the square roots of 169, 8281, 1221, 1234321; 62453; 91826; 953·26; 6·5; ·12, to 3 places of decimals.

* The reasons for these rules of evolution are best explained with the aid of algebraical symbols.

II. Extract the cube root of 1728, 12321, 753571; 6253210625; 83; 91·26; 6342·1, ·0001, to 2 places of decimals.

PART II.—THE ARITHMETIC OF CONCRETE QUANTITIES.

Fundamental Rules.

67. All the magnitudes which were considered in the first part of this work were numbers expressed according to the *decimal* system of notation; and the value of the figures increased from right to left according to the *powers* of 10. We shall now treat of expressions, the several terms of which increase in value from right to left, according to a variety of systems of notation.

68. Take, for instance, the concrete quantity £629. 17s. 8¾d.; that is, "six hundred and twenty-nine pounds seventeen shillings and eightpence three farthings." Here the 629 and the 17 separately obey the law of the decimal notation; but we are not to suppose that the 9 is worth ten times more than if it held the place of the 1; or the 7 ten times more than if it held the place of the 8. The 9 represents "nine pounds," and the 1 represents "ten shillings;" and as one pound contains twenty shillings, the 9 is by its position worth twice as much as if it held the place of the 1, and *twenty* times as much as if it held the place of the seven. So the 8 represents "eight pence," and as one shilling contains twelve pence, the 7 represents *twelve* times as much as if it held the place of the 8. Again, the ¾d. is expressed in another system of notation, namely, that of ratio fractions. The 4 signifies that we have taken a number of "farthings," and the 3 denotes that the number of farthings taken is three. Hence the 8 represents *four* times as much as if it held the place of the 3. Thus, in order to be able to read and make use of such concrete expressions as £629. 17s. 8¾d.,

12 yards 2 feet 6 inches, 1 ton 15 hundredweight 3 quarters 22 pounds, it is necessary to be acquainted, not only with the decimal system, but also with the system of money values, weights, or measures in which the concrete terms are expressed.

69. When these systems are known, the processes for addition or subtraction are the same as with abstract numbers. Thus—

£	*s.*	*d.*	
629	17	$8\frac{3}{4}$	
85	12	$10\frac{1}{2}$	
94	16	$9\frac{1}{4}$	
810	7	$4\frac{1}{2}$	—sum.

$$20)47(2 \quad \frac{40}{7} \qquad 12)28(2 \quad \frac{24}{4} \qquad 4)6(1 \quad \frac{4}{2}$$

£	*s.*	*d.*	
815	12	$8\frac{1}{4}$	
98	16	$8\frac{1}{2}$	
716	15	$11\frac{3}{4}$	—remainder.

3	12	20	6
2	7	9	3
1	5	11	3

70. It is not usual to multiply one concrete expression by another. Strictly speaking, indeed, we can only multiply by abstract numbers; but there are certain cases in which concrete quantities are considered as being multiplied together, and in which a special process is employed for that purpose. It will, however, always be possible to reduce a concrete expression to a single term—as, for instance, a sum of money to farthings; and the number denoting the value of the expression in this single denomination may be considered as *abstract,* and used for a multiplier.

71. Although we may reasonably speak of dividing one concrete quantity by another, still it is both simpler and more usual to reduce the divisor to the form of an abstract number, as just explained. The methods of reduction will be stated below. We will now give the rules for the multiplication and division of concrete quantities by abstract numbers.

72. RULE.—*Multiplication. If the multiplier is small,* set it under the number of the lowest denomination (on

the *right)*, and multiply this denomination. If the product reaches the number which connects this denomination with the one above it,* divide the product by this connecting number, set down the remainder in the first denomination, and carry the quotient to the next. Now multiply the second denomination by the multiplier, add in the quotient carried, and proceed as before. Thus—

tons.	(20) cwt.	(4) qrs.	(28) lbs.
2634	13	3	25
			9
23712	5	3	1

$$\begin{array}{r} 20)125(6 \\ 120 \\ \hline 5 \end{array} \qquad \begin{array}{r} 4)35(8 \\ 32 \\ \hline 3 \end{array} \qquad \begin{array}{r} 28)225(8 \\ 224 \\ \hline 1 \end{array}$$

If the multiplier be large, we may break it up into factors, and multiply by these successively. Or, if it be a prime, we may take the nearest number to it which is composed of factors, multiply by these factors successively, and correct the result by adding or subtracting. Thus—

yds.	ft.	in.	
37	2	11	
		6	$\times\ 9 - 1 = 53$
227	2	6	
		9	
2044	1	6	
37	2	11	
2006	1	7	

73. With very large multipliers, the following method will obviate the necessity of finding factors. Multiply—

* Four farthings = 1 penny; then 4 is the number which connects the denominations of farthings and pence.

£	s.	d.	
629	17	8	by 15685.
		10	
6298	16	8	= ten times.
		10	
62988	6	8	= one hundred times.
		10	
629883	6	8	= one thousand times.
		10	
6298833	6	8	= 10,000 times.
3149416	13	4	= 5,000 „
377930	0	0	= 600 „
50390	13	4	= 80 „
3149	8	4	= 5 „
9879720	1	8	= 15,685 „

74. Rule.—*Division.* If the divisor is small, employ the short process of division; if the divisor is large, employ the long process (§ 9). Set the divisor on the left of the dividend, and divide the highest denomination. If there is any remainder, reduce it to the next lower denomination, by multiplying it by the connecting number, and add in the given term of this lower denomination. Then divide again, and proceed in the like manner. Thus—

```
      £        s.     d.
 137)629       17     8¾(£4
     548
      81                    137)1568(11d.
      20                        137
137)1637(11s.                    198
    137                          137
     267                          61
     137                           4
     130                    137)247(1 f
      12                        137
    1568                        110
```

The answer is £4. 11s. $11\frac{1}{4}$d. + $\frac{119}{137}$ f.; or, if we stop at the pence denomination, which is more usual, £4. 11s. $11\frac{61}{137}$d.

75. Most of the operations of ordinary commerce consist of simple applications of the four fundamental rules. The following cases will illustrate the abbreviated methods most frequently employed, so far as not to involve the principle of *proportion.*

CASE I.—What cost $11\frac{3}{4}$ tons, at £12. 15s. 8d. a ton?

£	*s.*	*d.*	£	*s.*	*d.*	
			12	15	8	
					11	
			140	12	4	(cost of 11 tons.)
4)38	7	0	9	11	9	(cost of $\frac{3}{4}$ ton.)
			150	4	1	(cost of $11\frac{3}{4}$ tons.)

CASE II.—What cost 532 articles, at £1. 6s. 8d.?

	£	*s.*	*d.*	
532 @ £1, =	532			
„ @ 6s. 8d. =	177	6	8	since 6s. 8d. = $\frac{1}{3}$ of £1.
„ @ £1. 6s. 8d. =	709	6	8	

CASE III.—What cost 532 articles, at £7. 6s. 10d.?

532 @ £1 =	532			
	7			
„ @ £7, =	3724			
„ @ £0. 6s. 8d. =	177	6	8	(that is, $\frac{1}{3}$ of £532.)
„ @ £0. 0s. 2d. =	4	8	8	(that is, $\frac{1}{40}$ of cost at 6s. 8d.)
„ @ £7. 6s. 8d. =	3905	15	4	

(Concrete measures of a concrete expression are called **aliquot parts.** Thus 6s. 8d. is an aliquot part of £1.)

CASE IV.—What cost 359 articles, at £3. 4s. 9d.?

	£		
	359		
	3		
	1077		
2s. 6d. = $\frac{1}{8}$ of £1.	44	17	6
1s. 3d. = $\frac{1}{2}$ of 2s. 6d.	22	8	9
1s. 0d. = $\frac{1}{20}$ of £1.	17	19	0
359 at £3. 4s. 9d. =	1162	5	3

CASE V.—What cost 1448 articles, at 17s. $9\frac{1}{2}$d.?

	£			
	1448			
	7			
	8)10136			
	1267			(since 17s. 6d = $\frac{7}{8}$ of £1.)
3d. = $\frac{1}{70}$ of 17s. 6d.	18	2	0	
$\frac{1}{2}$d. = $\frac{1}{6}$ of 3d.	3	0	4	
	1288	2	4	

CASE VI.—What cost 749 articles, at 19s. $10\frac{1}{2}$d.?

	£	s.	d.	
d.	749	0	0	at £1.
8)749 at $1\frac{1}{2}$d.	4	13	$7\frac{1}{2}$	
93s. + 5 × $1\frac{1}{2}$d.	744	6	$4\frac{1}{2}$	(by subtraction.)

CASE VII.—What cost 24 lb. 11 oz. 10 dwt., at £3. 4s. $7\frac{1}{4}$d. a lb.?

	£	s.	d.
	3	4	$7\frac{1}{4}$
			24
	77	10	6
6 oz. = $\frac{1}{2}$ of 1 lb.	1	12	$3\frac{5}{8}$
4 oz. = $\frac{1}{3}$ of 1 lb.	1	1	$6\frac{5}{12}$
1 oz. = $\frac{1}{4}$ of 4 oz.	0	5	$4\frac{29}{48}$
10 dwt. = $\frac{1}{2}$ of 1 oz.	0	2	$8\frac{29}{96}$
	£80	12	$4\frac{91}{96}$

CASE VIII.—What cost 13 acres, 2 roods, 38 poles, at £3. 2s. 10d. a rood?

(Reduce 13 acres, 2 roods, to roods, and proceed as above.)

The methods indicated in the preceding cases are sometimes called by the meaningless name of **Practice**.

CASE IX.—Make out a bill of the following articles:—25 dozen of port, at 54s. a dozen; 30 dozen sherry, at 63s.; 15 dozen claret, at 21s.; 3 dozen brandy, at 52s.; 13 boxes cigars, at £1. 2s. 6d.; charging bottles at 1d. each.

A. B., *Dr.* to C. D.

	£	*s.*	*d.*
To 25 dozen Port, @ 54/,............	82	10	0
,, 30 ,, Sherry, @ 63/,............	94	10	0
,, 15 ,, Claret, @ 21/,............	15	15	0
,, 3 ,, Brandy, @ 52/,............	7	16	0
,, 13 boxes manillas, @ £1. 2s. 6d.,...	14	12	6
,, 73 dozen bottles, @ 1d., returnable,	3	13	0
£	218	16	6

Thus, **Bills of Parcels**, in their simplest form, involve ordinary multiplication and addition.

76. Tare is an allowance made on the sale of goods, for the weight of the packages, &c., in which they are sent.

Tret is an allowance for waste in the gross.

Draft is an allowance made by wholesale dealers, to retailers, to compensate for waste in making up small packages.

Either, or all of these, being subtracted from the *gross* weight, that which remains is called the **net** weight.

Arithmetical Tables.

77. The following are the Tables of Weights and Measures most frequently in use; and, with the Multiplication Table of the first twelve numbers, they should

be thoroughly committed to memory before the student proceeds any further.

MONEY.

4 farthings	=	1 penny.	4 f., or 4 q.	=	1d.
12 pence	=	1 shilling.	12d.	=	1s.
2 shillings	=	1 florin.	2s.	=	1fl.
20 shillings	=	1 pound.	20s.	=	1*l* or £1.
5 shillings	=	1 crown.	5s.	=	1 cr.
21 shillings	=	1 guinea.	21s.	=	1 gu.

TROY WEIGHT.

(Used for Gold, Silver, Jewels, and Scientific weighing.)

24 grains	=	1 pennyweight.	24 grs.	=	1 dwt.
20 pennyweights	=	1 ounce.	20 dwt.	=	1 oz.
12 ounces	=	1 pound.	12 oz.	=	1 lb.
4 grains	=	1 carat.			

AVOIRDUPOIS WEIGHT.

(For all heavy goods.)

16 drams	=	1 ounce.	16 drs.	=	1 oz.
16 ounces	=	1 pound.	16 oz.	=	1 lb.
28 pounds	=	1 quarter.	28 lbs.	=	1 qr.
4 quarters	=	1 hundredweight.	4 qrs.	=	1 cwt.
20 hundredweight	=	1 ton.	20 cwt.	=	1 ton.
14 pounds	=	1 stone.			
112 pounds	=	1 cwt.			
1 lb. Avoirdupois	=	7000 grains Troy (by statute).			
1 lb. Troy	=	5760 grains Troy.			

Thus 1 lb. Troy = $\frac{576}{700}$ lb. Avoir. = $\frac{144}{175}$ lb. Avoir.

APOTHECARIES' WEIGHT.

20 grains	=	1 scruple.	20 grs.	=	1 ℈.
3 scruples	=	1 dram.	3 ℈	=	1 ʒ.
8 drams	=	1 ounce.	8 ʒ	=	1 ℥.
12 ounces	=	1 pound.			

(The grain, ounce, and pound, are the same as in Troy.)

(For Fluids).

60 minims	=	1 dram.
20 ounces	=	1 pint *in measure*.

LENGTH.

12 inches	=	1 foot.	12 in.	=	1 ft.
3 feet	=	1 yard.	3 ft.	=	1 yd.
$5\frac{1}{2}$ yards	=	1 pole.	$5\frac{1}{2}$ yds.	=	1 p.
40 poles or 220 yards	=	1 furlong.	220 yds.	=	1 furl.
8 furlongs	=	1 mile.	8 furl.	=	1 m.

12 lines	=	1 inch.
4 inches	=	1 hand.
6 feet	=	1 fathom.
1760 yards	=	1 mile.
3 miles	=	1 league.
100 links or 66 feet	=	1 chain of land.

(In Geographical Measure, 60 minutes = 1 degree, 360 degrees = the circumference of a circle. Also 1 minute = 1 Geographical Mile.)

CLOTH MEASURE.

$2\frac{1}{4}$ inches	=	1 nail.	$2\frac{1}{4}$ in.	=	1 nl.
4 nails	=	1 quarter.	4 nls.	=	1 qr.
4 quarters	=	1 yard.	4 qrs.	=	1 yd.
3 quarters	=	1 Flemish ell.	3 qrs.	=	1 Fl. e.
5 quarters	=	1 English ell.	5 qrs.	=	1 E. e.
6 quarters	=	1 French ell.	6 qrs.	=	1 Fr. e.

SQUARE MEASURE.

(Used chiefly for land.)

144 sq. inches	=	1 sq. foot.
9 sq. feet	=	1 sq. yard.
$30\frac{1}{4}$ sq. yds.	=	1 sq. pole.
40 sq. poles	=	1 rood.
4 roods	=	1 acre.

CUBIC MEASURE.

1728 c. inches	=	1 c. foot.
27 c. feet	=	1 c. yard.

CAPACITY.

2, or 4 gills	=	1 pint.
2 pints	=	1 quart.
4 quarts	=	1 gallon.
2 gallons	=	1 peck.
4 pecks	=	1 bushel.
8 bushels	=	1 quarter.

---

42 gallons	=	1 tierce, *of wine*, &c.	
63 „	=	1 hogshead „	1 hhd.
84 „	=	1 puncheon „	
126 „	=	1 pipe or butt *of wine*, &c.	
252 „	=	1 tun. „	

78. Examples (not involving Proportion.)

I.

A. £	s.	d.	B. £	s.	d.	C.
16913285	13	$8\frac{1}{2}$	69432	13	$4\frac{1}{2}$	3, 18, 256
9945623	12	$9\frac{1}{4}$	38964	12	$9\frac{1}{4}$	7, 729
8863428	17	$11\frac{3}{4}$	18325	17	$6\frac{1}{2}$	9, 1728
18125694	8	2	89546	8	$9\frac{3}{4}$	12, 512
9954364	19	$8\frac{3}{4}$	72895	12	$8\frac{1}{2}$	63, 5760
9856932	16	$9\frac{1}{2}$	6184	8	$10\frac{1}{4}$	8913
9526329	12	$6\frac{3}{4}$	8632	19	$11\frac{3}{4}$	3295
8719285	11	$2\frac{1}{4}$	69165	13	$9\frac{1}{2}$	37243

tons.	cwts.	qrs.	lbs.	oz.	drs.	qrs.	lbs.	oz.	drs.	
516	13	3	21	12	8	3	24	15	12	8, 9, 12
272	4	1	3	4	5	2	20	12	8	12, 15, 25
187	6	2	5	8	2	3	19	7	7	64, 144, 576
96	5	1	8	5	7	2	9	10	10	1234

miles.	furl.	poles.	yds.	ft.	ft.	p.	yds.	ft.	
1639	7	39	5	2	6	39	5¼	1	11, 54, 72
876	6	25	4	1	7	30	5	2	13, 22
763	5	22	3	2	6	25	4	2	136
qrs.	bush.	pks.	gals.	qts.	bush.	pks.	gals.	qts.	
126	6	2	1	1	7	3	1	3	27, 88
33	7	3	1	2	6	3	1	3	162
27	1	2	1	1	3	3	1	2	544

Add the quantities in A and B, two or more at a time. Subtract any in B from the opposite one in A; or any in A from the one above it.

Multiply or divide any in A or B by the opposite numbers in C. Reduce any in A or B to any single denomination. Prove the results, as in § 10.

II.—(1.) What cost 3 tons 2 qrs. 16 lbs., at £5 a ton, at 6d. a qr., at 3½d. a lb.?

(2.) What cost 153 tons 3 qrs. 21 lbs., at £2. 7s. 6d. a ton, at 6s. 6d. a lb.?

(3.) What cost 27 hhd. 22 gals. 2 qts., at £3. 12s. 8d. a hhd., at 4½d. a qt.?

(4.) What cost 36 tuns 1 pipe 1 hhd. 20 gals., at £27. 11s. 8d. a tun, at £3. 7s. 4d. a hhd., at 4d. a qt.?

(5.) What cost 3 acres 1 rood of land, at £1060 an acre, at £1. 15s. 6½d. a sq. yard?

(6.) Make out a bill for the following:—5 yds. 3 qrs. of cloth, at 7s. 4d. a yard; 15 cwt. 1 qr. of sugar, at 7½d. a lb.; 2 hhds. of wine, at 37s. a gallon; 47 qrs. of oats, at 27s. 6d. a quarter; 3 cwt. 2 qrs. 21 lbs. cheese, at 7¼d. a lb.?

(7.) What is the net weight of 5 chests, weighing 27 cwt. 2 qrs. 12 lbs. gross, tare 1 cwt. 2 qrs., tret 4 lb. per 108?

(More exercises in ordinary *Practice* may be had by using the preceding table; taking for the cost only one or two digits in the pounds column. Thus, What is the cost of 3 qrs. 24 lbs. 15 oz. 12 drs., at £2. 13s. 4½d. per qr.,—from column B.)

Proportion of Concrete Quantities.

79. The principles of proportion, as explained in § 41—48, may be employed in the solution of many arithmetical questions. The only fresh consideration introduced is how to state the terms of the proportion.

80. Suppose three quantities given, and a fourth required, which shall be in the same proportion with one of the given quantities, as the remaining two are with each other.

81. RULE.—Set x in the fourth place to represent the required term. Place the term with which x is compared (that is, the other term of that *ratio*), in the third place: Now, if the ratio of the *third to the fourth* is, from the nature of the question, a ratio of *greater inequality*,* arrange the first and second terms as a ratio of greater inequality (that is, set the *least* in the second place); but if the ratio of the third to the fourth is a ratio of *less* inequality, arrange these as a similar ratio.

82. In other words, if the fourth term (the answer), is to be *greater* than the third, the second must be *greater* than the first; if the fourth is to be *less* than the third, the second must be *less* than the first.

EXAMPLE.—If 42 cwt. 2 qrs. cost £73. 16s. what is the cost of 41 cwt. 20 lbs.?

Here the *answer* is to be *cost*. Put the given *cost* in the third place. The ratio of 3d : 4th will clearly be a ratio of *greater* inequality, or the 4th will be $<$ the 3d. Therefore the 2d must be $<$ the 1st. Hence—

cwt.	qrs.		cwt.	lbs.		£	s.		
42	2	:	41	20	: :	73	16	:	x
4			4						
170			164						
28			28						
1360			1332						
340			328						
4760			4612						

* That is, a ratio whose value is *greater than the ratio of equality*; or > 1; as 3 : 2 or $\frac{3}{2}$, (see § 48).

The first two terms having been brought to the same denomination (pounds' *weight*), the work will proceed according to § 46. But a shorter and preferable method is as follows:—

$$x = \frac{£73\tfrac{4}{5} \times 41\tfrac{5}{28}}{42\tfrac{1}{2}} = £\frac{369 \times 1153 \times \cancel{2}}{5 \times \cancel{28}_{14} \times 85}$$

$$= £\frac{425457}{5950} = £71.\ 10s.\ 1\tfrac{173}{595}d.$$

This method, by complex ratio fractions, should be employed whenever the first two terms of the proportion can readily be expressed as mixed numbers; and particularly when the reduction of the *second* term (a *multiplier*), would produce a large number.

Compounded Ratios and Compound Proportion.

83. A ratio is said to be **compounded** of two or more other ratios, when it is equal to the ratio between the product of all the first terms, and the product of all the second terms.

Thus, if we have the ratios $2:3$, $5:6$, $9:12$; the ratio compounded of these will be $2 \times 5 \times 9 : 3 \times 6 \times 12$.

84. Now, suppose a number of ratios to be given, and also an invariable "third" term of a proportion. Then we can find a "fourth" term of the proportion, such that the ratio of the "third" to the "fourth," shall be in the ratio compounded of any number of other ratios. We may exhibit the process by an example.

$$\left.\begin{array}{r} 2 : 3 \\ 5 : 6 \\ 9 : 12 \end{array}\right\} :: 8 : x$$

$$\therefore x = \frac{8 \times (3 \times 6 \times 12)}{(2 \times 5 \times 9)} = \frac{4 \times 2 \times 12}{5} = \frac{96}{5} = 19\tfrac{1}{5}.$$

85. Another example will show how this method may be applied to questions involving the compound proportion of concrete quantities.

If 10 men dig a trench 80 yds. long, 6 yds. wide, and 2 ft. 8 in. deep, in 6 days, how deep a trench can 96 men dig in 3 days, if it be 240 yds. long, and 16 yds. wide?

$$\left.\begin{array}{lrcr} \textit{men,} & 10 & : & 96 \\ \textit{length,} & 240 & : & 80 \\ \textit{width,} & 16 & : & 6 \\ \textit{days,} & 6 & : & 3 \end{array}\right\} \begin{array}{c} (\textit{depth.}) \\ \text{ft.}\quad \text{in.} \\ ::2 \quad 8 : x \\ (= \tfrac{8}{9}\text{ yd.}) \end{array}$$

$$\therefore x = \frac{8 \times \overset{2}{\cancel{96}} \times \cancel{80} \times \cancel{6} \times \cancel{3}}{\underset{5}{\cancel{9}} \times \underset{3}{\cancel{10}} \times \underset{2}{\cancel{240}} \times \cancel{16} \times \cancel{6}} = \frac{8}{15}\text{ yd.} = 1\text{ ft. }7\tfrac{1}{5}\text{ in.}$$

86. Rule.—Set the given term of the ratio to which the answer belongs in the third place. From the remaining conditions of the question take each ratio, or *couple of corresponding concrete quantities* separately, and treat it with the third term as a case of simple proportion. When all the first ratios have been stated, work out as above.

Per Centages.

87. The method of calculating *per centages* was explained in § 46. The application of the principle to concrete quantities presents no difficulty. The following are ordinary instances.

88. Commission is so much per cent. allowed to an agent on the money passing in any transaction.

Brokerage is so much per cent. allowed to an agent in the buying or selling of shares, &c.

89. Insurance is the payment of so much per cent. to the insurers or underwriters, on the sum of money insured under any risk. There is also a Government tax of so much per cent. on the same sum. (The sum paid to the insurers is called the **premium**; and the paper containing the agreement is called the **policy**.)

Life assurance is of the same nature.

90. Interest is so much per cent. allowed for the use of money. The money used is called the **principal**; the principal, with its interest after a given time, is called

the **amount** at that time; generally computed at periods of one year.

91. Compound interest is when the simple interest is periodically added on to the principal, instead of being withdrawn; interest being then reckoned on the successive amounts.

There are several other applications of the principle of *per centages* which will be explained hereafter.

92. Interest.

EXAMPLE I.—The interest on £1281. 1s. 6d., for three years, at $4\frac{1}{2}$ per cent.

$$\begin{array}{lllll} & \pounds & s. & d. & \\ \text{For 1 year, } 100 : & 1281 & 1 & 6 & :: 4\frac{1}{2} : x. \end{array}$$

$$x = \frac{\pounds 1281 \quad 1 \quad 6 \times 9}{100 \times 2} = \pounds 6.\ 16s.\ 2\tfrac{28}{50}d. \times 9.$$

$$= \pounds 61.\ 5s.\ 11\tfrac{11}{50}d.$$

$$\therefore \text{ Int. for 3 years} = \pounds 183.\ 17s.\ 9\tfrac{33}{50}d.$$

$$\text{Or, simply, Int.} = \frac{\text{Principal} \times \text{rate} \times \text{No. of years.}}{100}$$

EXAMPLE II.—Interest on £600 for 49 days, at 7 per cent. per annum.

$$\left.\begin{array}{lr} \textit{principals,} & 100 : 600 \\ \textit{time,} & 365 : \ \ 49 \end{array}\right\} :: \begin{array}{c} \textit{interest} \\ 7 : x \end{array}$$

$$\therefore x = \frac{\pounds 600 \times 49 \times 7}{100 \times 365} = \pounds 5.\ 12s.\ 9\tfrac{15}{73}d.$$

EXAMPLE III.—£700 in 5 years amounted to £720. 16s. 3d. (simple interest). What was the rate per cent.?

The interest for 5 years being	£20	16	3
That for 1 year is	4	3	3

$$700 : 100 :: \pounds 4.\ 3s.\ 3d. : x.$$

$$\therefore x = \frac{100 \times \pounds 4 \quad 3 \quad 3}{700} = 11s.\ 10\tfrac{1}{2}d. \text{ per } \pounds 100.$$

Discount and Present Value.

93. The **present value** of a sum due at a future time is such a sum as, put out to interest, would, in the interval, amount to the sum due. The difference between the present value, and the sum due—in other words, the *interest on the present* value, is the *true discount.*

94. If a sum has been promised, and payment guaranteed by a bill falling due on a given date, the law allows **three days of grace** beyond the said date, not including Sundays.

95. The **usual discount** is a per centage on the *sum due* at a future date, calculated for the time between the pre-payment and the maturing of the debt, bill, &c.

EXAMPLE I.—The present value of £6060, at 3 per cent., due in 219 days.

$$365 : 219 :: 3 : x \quad \text{(Int. on £100 for 219 days.)}$$

$$x = \tfrac{219 \times 3}{365} = £\tfrac{9}{5} = £1.\ 16s.\ 0d.$$

Thus, the P. V. of £101. 16s. 0d., due in 219 days, = £100.

$$\begin{array}{l} \quad\ \ £ \quad s. \qquad £ \qquad\ \ £ \\ \therefore 101\ \ 16 : 6060 :: 100 : x \text{ (P. V. of £6060).} \end{array}$$

$$x = £\frac{6060 \times 100}{101\frac{4}{5}} = £\frac{6060 \times 100 \times 5}{509}.$$

EXAMPLE II.—A bill is drawn June 3, at 3 months, for £525, and paid August 16. What discount must be allowed to the payer, at $3\frac{1}{2}$ per cent.?

(1.) Find extent of pre-payment. The bill is due September 6, and is therefore paid 21 days before maturity.

(2.) Find of what sum £100 is the P. V. First find Int. on £100 for 21 days.

$$365 : 21 :: 3\tfrac{1}{2} : x$$

$$\therefore x = \tfrac{21 \times 7}{365 \times 2} = £\tfrac{147}{730}.$$

$$\therefore £100 \text{ is the P. V. of } £100 + £\tfrac{147}{730} = £\tfrac{73147}{730}.$$

(3.) Find P. V. of £525.

$$\tfrac{73147}{730} : 525 :: 100 : x.$$

$$x = £\tfrac{52500 \times 730}{73147} = £523.\ 18s.\ 10\tfrac{57338}{73147}d.$$

(4.) Thus, the true discount = (sum due) − (P. V.)

$$= £1.\ 1s.\ 1\tfrac{15809}{73147}d.$$

96. **Annuities** are sums paid periodically, and guaranteed either for a certain number of payments, or annually (quarterly, &c.), during the lifetime of the recipient.

The sum to be paid for this benefit will clearly be the present value of the several sums payable at given future dates.

Thus, if a life be reckoned at the estimate of 6 years, and the annuity to be purchased be £50—

The P.V. + int. for 6 years
= 6 × £50 + int. on £50 for 5 years, 4 years, 3 years, 2 years, 1 year.
= £300 + int. on £50 for 15 years (simple interest.)
which at 5 per cent. = £300 + £37. 10s. = £337. 10s.

The student is referred to the further consideration of annuities, in the Appendix to the *Algebra*.

Stocks and Shares.

97. **Stock** is the nominal amount of the capital of a company, divided into **shares**, which are represented by **scrip**.

98. As both the value of money and the credit of companies vary, the nominal price of shares is generally different from their actual price. Thus, a share of £100, and bearing interest at the rate of £100, may be bought for less or more than £100, as the case may be. When £100 share is worth £100, the stock is said to be **at par**; when more, **at premium**; when less, **at discount.** "**The Stocks**" usually means the government securities, and

represents the funded debt, interest on which is guaranteed by the national credit.

99. Stocks are of different kinds, and bear different rates of interest. Thus, we speak of the 3 per cents, the 4 per cents, &c.

100. Stock-brokers charge a commission of $\frac{1}{8}$ per cent. on the *stock* (2s. 6d. in £100), for negotiating the sale and purchase of stocks. The price of stocks must, therefore be increased to the buyer, and diminished to the seller, in this proportion.

EXAMPLE I.—I sell out (for myself) £5000 at 75. What do I get for it?

$$100 : 75 :: 5000 : x.$$

EXAMPLE II.—I employ a broker to sell out £1250 from the 4 per cents at 120, and buy into the $3\frac{1}{2}$ per cents at 95. What is the change in my income?

(1.) Find his late income.

$$100 : 1250 :: 4 : x; \quad \therefore x = £\tfrac{5000}{100} = £50.$$

(2.) Find the selling price of £1250 at 120.

$$100 : 120 :: 1250 : x; \quad \therefore x = £\frac{12 \times 125}{1} = £1500.$$

Broker's commission $= 12\frac{1}{2} \times$ 2s. 6d. = £1. 11s. 3d.

$\therefore$ sum realized = £1498. 8s. 9d.; (or, £1500 − £$\frac{25}{16}$ = £1498$\frac{7}{16}$.)

(3.) Find the amount in the $3\frac{1}{2}$ per cents at 95 of £1498$\frac{7}{16}$.

$$95\tfrac{1}{8} : 100 :: £1498\tfrac{7}{16} : x.$$

$$x = £\frac{23975 \times \overset{50}{\cancel{100}} \times \cancel{8}}{761 \times \cancel{16}} = £1575.\ 4s.\ 7\tfrac{144}{761}d.$$

(4.) Find the int. on £1575$\frac{175}{761}$ at $3\frac{1}{2}$ per cent.

(5.) Subtract his late income from his future one.

101. The following are examples of several classes of questions, involving the principle of proportion, which will frequently be met with.

I. *(Profit and Loss.)* I bought wheat at 60s. a quarter, and was obliged to sell it at 56s. What was my loss on the whole quantity?

The loss being 4s. on 60s.

$$60 : 4 :: 100 : x,$$

and $x = \frac{400}{60} = \frac{40}{6} = 8\frac{1}{3}$ per cent.

II. *(Profit and Loss.)* I buy goods at 45s. a dozen, and want to make 10 per cent. profit. At what price must I retail them?

$$100 : 110 :: 45\text{s.} : x$$

$$\therefore x = \frac{110 \times 45}{100}\text{s.} = \frac{990}{20}\text{s.} = 49\text{s. } 6\text{d.}$$

the selling price per dozen.

The price of each article = 4s. $1\frac{1}{2}$d.

III. *(Profit and Loss.)* A tradesman makes 30 per cent. profit by selling soda-water at 2s. a dozen. What did the goods cost him?

$$130 : 100 :: 2\text{s.} : x$$

$$x = \frac{200}{130}\text{s.} = 1\frac{7}{13}\text{s.} = 1\text{s. } 6\frac{6}{13}\text{d.}$$

IV. *(Profit and Loss.)* By selling wine at 24s. a dozen I lost $\frac{1}{2}$ per cent. At what must I sell it to gain 10 per cent.?

$99\frac{1}{2} : 100 :: 24\text{s.} : x$, the cost price per dozen.

$$\therefore x = \frac{24 \times 100}{99\frac{1}{2}}\text{s.}$$

Again,

$100 : 110 :: \frac{24 \times 100}{99\frac{1}{2}}\text{s.} : y$, the new selling price.

$$\therefore y = \frac{24 \times 100 \times 110}{99\frac{1}{2} \times 100}\text{s.} = \frac{24 \times 110}{99\frac{1}{2}}\text{s.}$$

Comparing these two statements, it is clear that we might have stated at once,

$$99\frac{1}{2} : 110 :: 24\text{s.} : x$$

$$\therefore x = \frac{110 \times 24 \times 2}{199}\text{s.} = \frac{5280}{199}\text{s.} = 26\frac{106}{199}\text{s.}$$

V. (*Profit and Loss.*) My customers ask for 5 per cent. discount? What must I charge sugar, which cost me 4d. a pound, to gain 15 per cent.?

100 : 115 :: 4d. : x, the selling price without discount.
$\therefore x = \frac{115 \times 4}{100}$d.

95 : 100 :: $\frac{115 \times 4}{100}$d. : y, the selling price with discount.

$$\therefore y = \frac{115 \times 4 \times 100}{100 \times 95}\text{d.} = \frac{115 \times 4}{95}\text{d.}$$

Hence it is clear that we might at once have stated,

$$95 : 115 :: 4\text{d.} : x.$$

$$\therefore x = \frac{115 \times 4}{95}\text{d.} = \frac{23 \times 4}{19}\text{d.} = \frac{92}{19}\text{d.} = 4\tfrac{16}{19}\text{d.}$$

VI. (*Proportional Parts.*) A, B, and C, invest in a speculation, £60, £80, and £100 respectively. The speculation gains £100. How must this profit be divided?

The three shares are in proportion to 60, 80, and 100.

Therefore, A's share $= \frac{60}{60+80+100} = \frac{6}{24} = \frac{1}{4}$ of the whole;
„ B's „ $= \frac{8}{24} = \frac{1}{3}$ „
„ C's „ $= \frac{10}{24} = \frac{5}{12}$ „
∴ A must receive $\frac{1}{4}$ of £100 = £25;
B „ $\frac{1}{3}$ of £100 = £33. 6s. 8d.;
C „ $\frac{5}{12}$ of £100 = £41. 13s. 4d.

VII. (*Proportional Parts.*) A, B, and C, were engaged on piece-work, for which they were to receive £6. A works twice as long as B, but only $\frac{1}{3}$ as long as C. What must each receive?

Their work (and pay) is in proportion to 2, 1, and 6.

Therefore, A will receive $\frac{2}{9}$ of £6.
B „ $\frac{1}{9}$ of £6.
C „ $\frac{2}{3}$ of £6.

VIII. (*Proportional Parts.*) A invests £50 for 12 months; B invests £60 for 9 months; C invests £80 for 6 months. How are the profits to be divided?

How much ought B to invest for 12 months, so as to make it equivalent to his £60 for 9 months?

$$12 : 9 :: £60 : x$$

$$\therefore x = £\frac{60 \times 9}{12} = £45.$$

Similarly, C's equivalent for 12 months $= £\frac{80 \times 6}{12} = £40$.
Hence, their shares are proportional to 50, 45, 40; that is, to 10, 9, 8.

Therefore, A receives $\frac{10}{27}$; B, $\frac{9}{27}$; C, $\frac{8}{27}$.

It is clear that the shares would be proportional to the products of the capitals by the times; that is, to 50×12, 60×9, and 80×6.

Duodecimals.

102. Duodecimals are sometimes adopted as a scale of notation, in place of decimals. Thus, 6394 in duodecimals $= 6 \times 12^3 + 3 \times 12^2 + 9 \times 12 + 4 = 10912$ in decimals.

But the term is more usually applied to the measurement by feet, inches, seconds, thirds, &c.; according to the following table:—

1 foot	= 12 inches	= 12′	(also called **primes.**)
1 inch	= 12 seconds	= 12″	
1 second	= 12 thirds	= 12‴	
1 third	= 12 fourths	= 12^{iv}, &c.	

Any concrete quantity in this measurement is considered as feet, and duodecimal parts of a foot; and two such quantities may be multiplied together in the following manner.

Multiply first by the feet of the multiplier, beginning with the lowest denomination in the multiplicand. Then multiply by the primes of the multiplier, setting down the results one place to the right of the corresponding denominations in the multiplicand (because the final product will be an expression in square measurement). Multiply by the seconds, &c., in the same manner; and add up the partial products.

(It will be seen that if any two denominations be multiplied, the sign of the denomination of the product will be the sum of the signs of the factors. Thus, $8^{iii} \times 11^{iv} = 88^{vii} = 7^{vi}\,4^{vii}$.)

EXAMPLE—

ft.	′	″		
32	6	11		
25	5	4		
814	4	11		
13	6	10	7	
	10	10	3	8
828 f.	10′	7″	10‴	8^{iv}

103. Duodecimal multiplication is mainly employed in computing the superficial measure and solid content of timber, and artificers' work. We add a simple example, solved by two methods.

What is the surface of a floor 15 ft. $3\frac{1}{2}$ in., by 18 ft. $9\frac{1}{2}$ in.?

ft.	′	″		
18	9	6		
15	3	6		
281	10	6		
4	8	4	6	
	9	4	9	0
287	4	3	3	

Otherwise, 15 ft. $3\frac{1}{2}$ in. $\times$ 18 ft. $9\frac{1}{2}$ in.

$$= 15\tfrac{7}{24} \times 18\tfrac{19}{24} = \frac{367 \times 451}{24 \times 24} = 287\frac{205}{12 \times 12 \times 4};$$

$$\left[\begin{array}{l} \quad 367 \\ \;1835 \\ \;1478 \\ 4)\overline{165517} \\ 12)\overline{41379}—1 \\ \;\;12)\overline{3448}—3 \\ \qquad 287—4 \end{array} \right\} \text{T.R.} = 205. \Bigg]$$

and 4′ 3″ 3‴

$$= \frac{4}{12} + \frac{3}{12 \times 12} + \frac{3}{12 \times 12 \times 12}$$

$$= \frac{576 + 36 + 3}{12 \times 12 \times 12} = \frac{615}{12 \times 12 \times 12}$$

$$= \frac{205}{12 \times 12 \times 4}.$$

104. **Examples**—Proportion, &c.

(1.) Bought 125 lbs. for £4. 7s. 6d.; how much can I buy for £12. 10s.? and what will 1 qr. cost?

(2.) Sold 36 gals. for £12. 6s. 8d.; what shall I get for 2 hhds.? and how much must I sell for £20?

(3.) If I gain $4\frac{1}{2}$d. in the shilling, what do I gain on £56? and what on the £100?

(4.) How many yards of paper, 4 feet long, will cover a wall 7 feet by 3 yards? and how much will paper a room 8 feet high, 20 long, and 15 broad?

(5.) What is the tax on £26, at 4d. in the £? what on £525. 7s. 6d., at the rate of 2s. 8d. on £2. 10s.?

(6.) The wages of 12 men for 15 days, when £2. 8s. pays 8 men for 3 days?

(7.) A carrier took 1 cwt. 3 qrs. 125 miles for £2. 10s.; how much ought he to charge for 150 lbs., carried 150 miles?

(8.) The cost of keeping 1000 sheep for 50 days = £1250; what number of sheep would £100 keep for 365 days?

(9.) A cistern, 20 ft. × 12 ft. × 8 ft., contains 100 gallons, at 12s. 6d. a gallon; what is the value of a cistern 5 feet larger each way?

(10.) I had £12. 10s. interest on £560 in the bank for 3 months; what annual interest shall I have on £1000?

(11.) I lose 3d. in the shilling on £100, and gain $1\frac{1}{2}$d. in the shilling on £158; what is my whole gain or loss per cent.?

(12.) How much 3 per cents' stock can be bought for £625. 10s. at $87\frac{1}{2}$? and what will be the income derived?

(13.) What difference in income arises from selling out £5000 from 7 per cents at 195, and buying into the 3 per cents at 87, allowing $\frac{1}{2}$ per cent. commission?

(14.) Present value and discount on £640. 12s. 6d., due in 146 days, at $4\frac{1}{2}$ per cent.?

(15.) Present value and discount on £590. 13s. 4d., due 3 years hence, at $3\frac{1}{2}$ per cent.?

(16.) Compound interest on £125, at 10 per cent., for 5 years?

(17.) What is my gain per cent. by selling goods at 32s. per cwt., for which I gave ½d. per lb.?

(18.) How much wine, at 54s. a dozen, must I receive for 3 boxes of cigars at £1. 2s. 6d., so as to make a profit of 3 per cent.?

(19.) I lose 7½ per cent. by selling an article for 10s. 6d.; what did I give for it?

(20.) What is the solid content of a cistern 6 feet long, 3 ft. 2 in. broad, and 35½ in. deep?

METRICAL SYSTEMS.

NOTATION, as was explained in § 1, is the process of representing numbers by figures. This is effected by setting down the figures in a line, and estimating their value according to their relative position, right and left of each other. As we move a figure from right to left, its value increases by a regular multiple; the one commonly employed being *ten*. Thus, in the number 54321 (fifty-four thousand three hundred and twenty-one) the 2 is ten times as large as if it stood where the 1 is; the 3 is ten times as large as if it stood where the 2 is; and so on. This multiple is called the **base** of the system; and the system whose base is 10 is called the **decimal** system.

When we come to the notation of concrete quantities we might, if we chose, represent them in the same *metrical* manner. Thus "7 yards 2 feet 5 inches" might be represented as "725 inches," *if there were a regular base throughout,* and if one denomination (such as the *inches* above) were invariably regarded as the unit of that particular system. But as our systems of weights and measures are different, and as the bases vary in almost every case, we are clearly unable to adopt this very convenient method of representing concrete quantities. Thus "725 inches" does not stand for the same thing as "7 yards 2 feet 5 inches," because 2 feet are not *ten times* as much as 2 inches, and 7 yards are not *ten times* as much as 7 feet.

It has often been proposed to reduce all our systems of notation, both of abstract numbers and of concrete quantities, to one and the same system, having a common

base throughout. There would be many advantages in this uniform *metrical system;* and we will now proceed to explain what its nature would be. The French nation has already adopted such a system, with *ten* for the common base; and the most serviceable manner of illustrating the special features of a metrical system will be to explain that which is employed in France.

It may, however, be remarked that, in adopting *ten* as the common base, we should lose the very great advantage of being able to employ weights and measures which are *one-half* and *one-quarter* of others; whilst to employ any other base than *ten*, after it has been used for abstract numbers so long, would be practically impossible. Hence the reason why no country, except France, has thought it worth while to adopt a uniform metrical system—decimal or other.

The French Metrical Systems.

The following are the tables of money-value, weights and measures, now used in France:—

MONEY TABLE.

(Unit, one *franc.*)

10 centimes = 1 decime.
10 decimes = 1 franc.

(This table is manifestly insufficient, and therefore the following coins are also employed;—

5 centimes = 1 sou.
20 francs = 1 Napoleon.

A franc is worth about $9\frac{3}{4}$d. of English money. The precise value at any particular time is subject to arrangement amongst the money-changers. It is generally safe to reckon a hundred francs as worth £4. A decime is nearly worth an English penny, and will as a rule pass as such in England.)

EXAMPLE OF NOTATION.—

357 centimes
= 3 francs 5 decimes 7 centimes.*

METRIC TABLE OF WEIGHT.

(Unit, one *gram*, or *gramme*.)

				English Grains.
10 milligrams	= 1 centigram	=		0·15432348
10 centigrams	= 1 decigram	=		1·5432348
10 decigrams	= 1 gram†	=		15·432348
10 grams	= 1 dekagram	=		154·32348
10 dekagrams	= 1 hectogram	=		1543·2348
10 hectograms	= 1 kilogram	=		15432·348
10 kilograms	= 1 myriagram	=		154323·48

(The prefixes to the word *gram* in the seven words above given are derived from the Latin and Greek, and express *ten*, and multiples of ten.

Thus *deci*, Latin, and *deka*, Greek, denote *ten;*
centi, Latin, and *hecto*, Greek, denote *a hundred;*
milli, Latin, and *kilo*, Greek, denote *a thousand;*
myria, Greek, denotes *ten thousand.*

The Latin prefixes are used to express fractional parts of the unit; and the Greek prefixes to express integral multiples of the unit. A *milligram* is one-thousandth of a gram; a *myriagram* is ten thousand grams.

* From this example it will be at once manifest that the process of *reduction* of concrete quantities disappears from the metrical system. All that it is necessary to know is the value of the *unit* in any number before us: thus,

357 centimes = 3 francs 5 decimes 7 centimes;
35 decimes = 3 francs 5 decimes;
57 centimes = 5 decimes 7 centimes.

† A *gram* is the weight of one cubic centimetre of distilled water.

The same prefixes are employed in the remaining French tables; therefore, when these are once mastered, no further difficulty will be met with.

The column on the right hand of the table will illustrate the great simplicity of a complete metrical system, in which both abstract and concrete quantities can be expressed upon a common base of notation.)

EXAMPLE OF NOTATION:—69342158 milligrams
= 6 myriagrams, 9 kilog., 3 hectog., 4 dekag.,
2 gr., 1 decig., 5 centig., 8 millig.

METRIC TABLE OF LENGTH.

(Unit, one *metre.*)

				English Inches.
10 millimetres	=	1 centimetre	=	·393708
10 centimetres	=	1 decimetre	=	3·93708
10 decimetres	=	1 metre	=	39·3708
10 metres	=	1 dekametre	=	393·708
10 dekametres	=	1 hectometre	=	3937·08
10 hectometres	=	1 kilometre	=	39370·8
10 kilometres	=	1 myriametre	=	393708

(One metre is about $3\frac{1}{3}$ inches longer than the English yard. One kilometre is nearly five furlongs.

Though the metre is theoretically the unit, long distances are usually reckoned by kilometres. In land-surveying the dekametre is found most convenient. The French chain, divided into fifty links, is one dekametre in length. Thus 1 link = 2 decimetres = about 7·9 inches.)

EXAMPLE OF NOTATION:—89307·541 metres
= 8 myriam., 9 kilom., 3 hectom., 7 metres,
5 decim., 4 centim., 1 millim.

As our number contained an integer and a decimal fraction, the word "metres" served to define the integral unit 7 as a number of metres.

Metric Table of Square Measure.

(Unit, one *are*.)

			Square Yards.
10 centiares	= 1 deciare	=	11·96033
10 deciares	= 1 are	=	119·6033
10 ares	= 1 dekare	=	1196·033
10 dekares	= 1 hectare	=	11960·33

(One acre = $\frac{2}{5}$ French hectare.

The hectare is generally employed in the measurement of land.

The *are* is one *square dekametre;* that is, in English measure, 393·7 inches × 393·7 inches.

It is a defect in the French system that the unit of square measure is not the square of the unit of linear measure.)

Example of Notation:—1387·952 ares
= 13 hectares, 8 dekar., 7 ar., 9 deciar., 5·2 centiar.

Here it will be seen that the original ·052 ares, requiring to be written as centiares, become 5·2; the unit being made 100 times less.

Example of Work.—"*A rectangular field measures 1 hectom. 3 dekam. 5 m. by 7 dekam. 4 m. What is its area?*"

Multiply 135 metres by 74 metres. The result is 9,990 *square metres*. Now, in one square dekametre there are 100 square metres: thus,

9,990 square metres = 99 sq. dekam.,
90 square metres = 99 ares 9 deciares.

$$\begin{array}{r} 135 \\ 74 \\ \hline 540 \\ 945 \\ \hline 9990 \end{array}$$

Metric Table of Cubic Measure.

(Unit, one *Stere*.)

10 centisteres	= 1 decistere	=	3·5317	cubic feet.
10 decisteres	= 1 stere	=	35·317	cubic feet.
10 steres	= 1 dekastere	=	353·17	cubic feet.

(The stere is a cubic metre; that is, the cube of 39·3708 inches = 35·317 cubic feet, *approximately*.)

METRIC TABLE OF CAPACITY (FOR LIQUIDS).

(Unit, one *litre*.)

			English Measure. Pints.		Cubic Equivalent.
10 centilitres	= 1 decilitre	=	·1760773	=	10 c. centim.
10 decilitres	= 1 litre	=	1·760773	=	1 c. decim.
10 litres	= 1 dekalitre	=	17·60773	=	1 centistere
10 dekalitres	= 1 hectolitre	=	176·0773	=	1 decistere.
10 hectolitres	= 1 kilometre	=	1760·773	=	1 c. metre, *or* 1 stere.

Such is the French metrical system, which is of great theoretical beauty, and extremely simple in its use.

We shall here add a number of miscellaneous examples, the solution of which will serve to impress what has gone before upon the student's mind.

EXAMPLES ON THE METRICAL SYSTEM OF MONEY-VALUE, WEIGHTS, AND MEASURES.

1. Reduce 7634295 centimes to Napoleons and francs.
2. Reduce 359·6823 Napoleons to francs, decimes, and centimes.
3. What French money must I receive for £23. 6s. 4d., at the rate of 25 francs to the pound?
4. What English money must I receive for 69 francs 8 decimes, at the rate of 9¾d. to the franc?
5. Multiply 6·39 francs by 52, and reduce the answer to centimes.
6. Add 1 myriam., 5 kilom., 8 hectom.; 7 hectom, 4 dekam., 6 m., 8 decim.; 7 myriam., 9 hectom., 8 m.; 12 m., 7 decim., 8 centim., 6 millim.; 5 hectom., 7 dekam., 9 m.; 125 m.; 623 decim.
7. Multiply 86 kilom., 9 hectom., 3 dekam. by 9.
8. Multiply 12 dekag., 6 gr., 7 decig. by 7.
9. Multiply 9 dekast., 4 st., 8 decist. by 213.

10. Multiply 5 dekar., 9 ar., 7 deciar., by 423.
11. In one stere how many cubic inches?
12. In 923 myriag. how many decig.?
13. In 19 myriam., 4 hectom., 3 decim. how many millim., and how many centim.?
14. How many inches in 1000 kilometres?
15. Take 20 yards from 20 metres, expressing the result in inches.
16. How many sq. decim. in 50 ares?
17. How many cub. decim. in 25 steres?
18. Reduce 9·5 sq. dekam. to sq. metres.
19. Reduce 6 sq. millim. to sq. metres.
20. Reduce 9 sq. myriam. to sq. metres.
21. Reduce 2·25 sq. decim. to sq. metres.
22. Add the last four results.
23. An are of land is worth £20; what is the value of a sq. metre, and of a hectare?
24. How many deciar. in 20 sq. metres?
25. How many sq. metres in ten plots of 32·5 hectares each?
26. Reduce 9 sq. met. 51 sq. decim. 39 sq. centim. 92 sq. millim. to sq. metres.
27. Express 3256·8942 metres in every denomination of metrical linear measure.
28. Express 3256·8942 ares in every denomination of metrical square measure.
29. Express 3256·8942 steres in every denomination of metrical cubic measure.
30. Express 3256·8942 grams in every denomination of metrical weight.
31. A hectare of land is worth £1400; what is the value of an English acre?
32. How many pints in 300 litres?

Reduce the following to English measure:—

33. 12 dekag., 3 g., 5 decig.
34. 1236 g., 2 centig., 8 millig.
35. 10 kilol., 5 hectol., 8 dekal., 5 l.

36. 20 dekast., 5 st., 8 decist.
37. 5 myriam., 6 kilom., 8 hectom., 9 dekam., 8 m.
38. 923 hectar., 7 dekar., 2 ar., 6 deciar.
39. 32 ar., 7 deciar., 8 centiar.
40. 8324·625 ares.
41. 93425·8125 hectares.
42. 1·5 litres.
43. 12 myriag., 7 kilog., 9 hectog.
44. 323·725 grams.
45. 8 Napoleons, 5 francs.
46. 529·825 Napoleons.
47. How many litre bottles can be filled from 100 imperial pint bottles of wine?
48. Find the weight of a cubic metre of distilled water.
49. I drink $2\frac{1}{2}$ litres every day. How long will a gallon of wine last me?
50. A kilogram weighs 2·68 lbs. Troy. What is the weight of 1 quintal, which = 100 kilograms?
51. What is the weight avoirdupois of 1 millier, which is equal to 1000 kilograms?
52. A cistern measures 3 metres long, 2 metres broad, and 1 metre deep. How many hectolitres does it contain?
53. The cost of the contents of this cistern, at $1\frac{1}{2}$d. per litre?

Of other Metrical Systems.

The Binary System.

Many propositions have been made in different countries to secure a uniform metrical system, similar to that adopted in France.

The simplest of these in many respects is the **binary** system, in which the common base is *two*. The advantage of making each higher denomination exactly *twice* the value of the one below it would be very great; and in concrete arithmetic especially a practical benefit would arise from having a number of weights and measures enabling us to reckon the *half*, *quarter*, &c., of any given quantity.

The result of adopting this system would be to create a multiplicity of denominations. If our unit—or lowest denomination, were taken very small, as it would be necessary to take it, we should require a large number of multiples in order to reach the highest denomination. Thus, if our smallest weight were one grain, we should need a series of twenty-five weights in order to be able to weigh a ton.

The inconvenience of this large number of weights and measures would not be so great as that of having an increased number of coins of different value; but the greatest inconvenience would arise in applying the binary system to abstract numbers—quite apart from the consideration that it would be impossible to give up our present decimal system altogether.

We will give an example of a number written in the binary system of notation. It will be observed that each figure has double the value which it would have if it were one place further to the right.

$$\begin{aligned} &1000101101 \text{ in the binary system} \\ &= (2)^{10} + (2)^{6} + (2)^{4} + (2)^{3} + 1 \\ &= 1024 + 64 + 16 + 8 + 1 \\ &= 1113 \text{ in the decimal system.} \end{aligned}$$

The figures 1 and 0 would be the only figures which we could employ.

The Duodecimal System.

Others have proposed to make 12 the base of a system of notation. The principal advantage of this would be the fact that 12 has four measures—namely, 2, 3, 4, and 6; and we could therefore easily reckon a *half, third, fourth,* and *sixth* of the standard unit of weight, measure, and money-value.

The advantage of this is practically illustrated by the existence of coins of the value of six-pence, four-pence,

and three-pence, respectively the half, third part, and fourth part of a shilling.

We will give an example of a number written in the duodecimal system of notation, in order to explain the use of the system in arithmetical operations. We should require in this case two new *digits*, of the value of *ten* and *eleven*. These we may represent by the letters *t* and *e*.

$604t8e$ in the duodecimal system
$= 6(12)^6 + 4(12)^4 + 10(12)^3 + 8(12)^2 + 11$
$= 6 \times 1985884 + 4 \times 165888 + 10 \times 1728 + 8 \times 144 + 11$
$= 12597299$ in the decimal system.

The Pound and Mil System.

More than one system have been proposed for adoption in England, which would retain 10 for a base. We shall only describe one, which was recommended by a commission of practical and scientific men in the year 1853, for coinage alone.

The unit of value would remain as it is at present, *one pound;* and there would be three sub-multiples of this unit. One of these smaller coins is already in use amongst us—namely, the florin.

Decimal Money-Table.

(Unit, one *pound.*)

10 mils	= 1 cent	(10 m. = 1 c.)
10 cents	= 1 florin	(10 c. = 1 fl.)
10 florins	= 1 pound	(10 fl. = 1 l., or £1.)

The adoption of this table of values, and the coinage corresponding to it, would possess many advantages; one of which would be a coin slightly less in value than our present farthing, and which we might expect to come into more general use, thus encouraging the sale of smaller quantities of cheap commodities.

The disadvantage would be the loss of *half* and *quarter* coins.

The operations of Arithmetic would be greatly simplified. Of this we will add a few examples—

1. Express £25 as mils.

Since £1 = 1000 mils.
£25 = 25000 mils.

2. Express 69342 mils as a mixed sum.

£69. 3 fl. 4 c. 2 mils.

3. Divide £6354. 2 fl. 5 c. 6 m. by 1000.

The sum = £6354·256.
Therefore the quotient = £6·354256.
or = £6. 3 fl. 5 c. 4·256 m.

4. Express £15. 19s. $10\frac{3}{4}$d. in the metric system of coinage.

£15				£15
19s.	= $9\frac{1}{2}$ fl.	= 9·5 fl.	=	£ ·95
6d.	= $\frac{1}{4}$ fl.	= ·25 fl.	=	£ ·025
3d.	= $\frac{1}{2}$ of 6d.		=	£ ·0125
$1\frac{1}{2}$d.	= $\frac{1}{2}$ of 3d.		=	£ ·00625
$\frac{1}{4}$d.	= $\frac{1}{6}$ of $1\frac{1}{2}$d.		=	£ ·001041$\dot{6}$
∴ £15. 19s. $10\frac{3}{4}$d.			=	£15·994791$\dot{6}$

or, = £15. 9 fl. 9 c. 4·791$\dot{6}$ m.

5. Express £3. 6 fl. 5 c. 3 m. in the ordinary system of coinage.

£3			£3
6 fl.		=	12s.
5 c.	= $\frac{1}{2}$ fl.	=	1s.
2 m.	= $\frac{1}{25}$ of 5 c.	=	$0\frac{12}{25}$d.
1 m.	= $\frac{1}{2}$ of 2 m.	=	$0\frac{6}{25}$d.
∴ £3. 6 fl. 5 c. 3 m.		=	£3. 13s. $0\frac{18}{25}$d.

American Decimal Money Table.

(Unit, one *dollar*.)

10 mils = 1 cent.
10 cents = 1 dime.
10 dimes = 1 dollar (1 $.)
10 dollars = 1 Eagle (1 E.)

(A dollar is worth about 4s. 2d. English money.)

We add a few examples on the various systems which we have described, in order to impress them on the student's memory.

EXAMPLES ON VARIOUS SYSTEMS.

1. Express 629 in the binary system.
2. Express 823 in the duodecimal system.
3. Bring $3t4e1$ (duodecimal) into the decimal system.
4. Bring 101101101 (binary) into the decimal system.
5. Express 524 in a system having 6 as its base.
6. Express 69842 in a system having 100 as its base.
7. Express 91863 in a system having 99 as its base.
8. Express the following sums as £, fl., c., m.:— £6·932, £928·325, £2·6954, £6256·253.
9. Add the quantities in (8), and express the sum as a number of florins, of cents, and of mils.
10. Subtract the first quantity in (8) from the last, and express the difference as mils.
11. If a franc be worth 10d., how many francs are there in 50 American dollars?
12. How many French centimes in an English pound?
13. How many French centimes in an American Eagle?
14. Reduce £3. 4 fl. 3 m. to mils.
15. Reduce £6942·342 to florins and mils.
16. My income is £425·35. How many mils can I spend each day?
17. Twenty sheep cost £125·65. What is the cost of each?
18. A family eats 9·25 lbs. of meat every day. If meat sells at 4·7 cents a pound, what is the butcher's bill for three months?
19. I want to build a new house. I sell my old one for £629·35, and raise £805·5 on mortgage. I have then half the money required. What will my new house cost to build?
20. A tradesman's receipts for six days are 69 fl. 3 c.; £25·68; 9863 mils; £6. 8 fl. 3 c. 5 m.; 69328 mils; and 912 fl. 3 mils. What does he take in the week?

21. I go to New York with £500, and change my money into dollars. How many dollars have I?

22. An American brings 1000 dollars to Paris. How many francs will he have to spend, reckoning a dollar at 4s. 2d., and a franc at 10d.?

23. I owe 69342·68 mils. I have in cash £60, and a cheque for £30·345. What shall I have remaining after paying my debt?

24. If there were five denominations lower than the mil, how many of the smallest coin would be contained in the value of a French franc (10d.)?

25. What would be the cost in English money of painting the walls of a room 20 metres high, 40 metres long, and 36 metres broad, when the paint costs 1 fr. 48 c. the square metre?

26. Express £169. 19s. $7\frac{3}{4}$d. in the metric system of coinage; and then convert it by the same method into the American system.

27. The weight of a cubic decimetre of gold is 19·3617 Kilograms. What is the weight of a cubic metre?

28. A wine merchant buys 5268 hectolitres of wine for ten thousand francs. He sells one third of the quantity for one half of the whole cost price, and makes 20 per cent. profit on the remainder. How much does he gain in all?

GEOMETRY.

DEFINITIONS.

Dimensions.

1. Of **dimension**, or **measurement**, there are three modes—length, width, and height.*

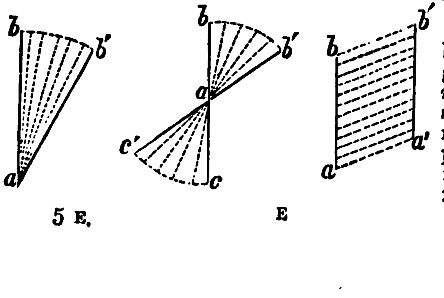
2. A **point** defines a position, and has no dimensions.

3. A **line** defines length only; or is measured in length only.†

4. A **surface** is measured in length and width only.‡

* These three dimensions correspond in some sense with the first three *powers* of a base (*Algebra*, § 5). Thus a might represent a certain *length;* a^2, a surface; a^3, a solid. Hence we call a^2, "a squared;" and a^3, "a cubed;" and we say that a, a^2, a^3, are of one, two, and three dimensions.

† A *line* may be considered as generated by the *motion of a point* from one position to another.



‡ A *surface* may be considered as generated by the *motion of a line* in such a manner that not more than one point in the generating line remains fixed.

5. A **solid** is measured in length, width, and height.*

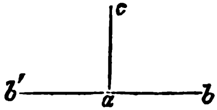
6. A **straight line** lies evenly between its extreme points.

Angles.

7. An **angle** is the inclination of two lines to each other.

8. A **rectilinear angle** is the inclination of two straight lines to each other.†

9. **Rectilinear right angles** are such that if one be placed adjacent and exterior to another, and one straight line of the one coincides with one straight line of the other, the two remaining lines are in one and the same straight line.



10. One straight line is **at right angles** to another, when the angles formed at the point of intersection are right angles.

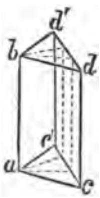
11. An **acute angle** is a less inclination than a right angle.

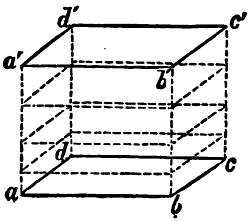
12. An **obtuse angle** is a greater inclination than a right angle.

Figures.

13. A **geometrical figure** is limited by boundaries.

* A *solid* may be considered as generated by the *motion of a surface* in such a manner that not more than one line in the generating surface remains fixed.





† This is a sufficient definition. The angle is measured at the point where the lines meet, or would meet. If the lines

14. A **line** is a figure limited by two points.*

15. A **superficial figure** is limited by one or more lines.

16. A **solid figure** is limited by one or more superficial figures.

Plane Figures.†

17. A surface is **plane** when, any two points being taken in it, the straight line joining them lies wholly in the surface.

18. A **rectilinear triangle** is a plane figure limited by three straight lines.‡

19. An **equilateral triangle** is limited by three equal straight lines.

20. An **isosceles triangle** has two sides equal.

21. A **scalene triangle** has its three sides unequal.

22. A **right angled triangle** contains a right angle.

23. An **acute angled triangle** has all its angles acute.

24. An **obtuse angled triangle** contains an obtuse angle.

Quadrilateral Figures.

25. A **square** has its sides equal, and its angles right angles.

26. A **rhombus** has its sides equal, but its angles are not right angles.

27. An **oblong** has its opposite sides equal, and its angles right angles.

meet and lie in one straight line, their inclination is said to be an angle of 180 degrees. If the lines never meet, they are assumed to meet "at infinity," and their inclination is 0.—(See *Trigonometry.*)

* These two points may coincide; when the line will be the circumference of a curvilinear figure.

† The geometry of plane figures is called geometry of two dimensions.

‡ One of these lines is called the base, and the other two the sides. The side opposite to the right angle in a right angled triangle is called the hypotenuse.

28. A **rhomboid** has its opposite sides equal, but its angles are not right angles.

Parallels.

29. **Parallel straight lines** are those which never meet, though produced continually in both directions.

30. A **parallelogram** is a four-sided figure, whose opposite sides are parallel.

31. Any four-sided figure which is not a parallelogram is called a **trapezium.***

32. The straight line joining two opposite angles of a quadrilateral is called a **diagonal.**

33. **Polygons,** or multilateral figures, are limited by more than four straight lines.

34. A **circle** is a figure contained by one line, called the **circumference,** and is such that all straight lines drawn from a certain point within the circle to the circumference are equal to each other.

35. This point is called the **centre** of the circle; and the straight lines drawn from the centre to the circumference are called **radiuses.**†

36. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

37. When a parallelogram is subdivided into four parts by straight lines parallel to the sides and intersecting in a diagonal, the two parts through which the diagonal passes are called the parallelograms about the diagonal, and the remaining parts are the **complements.**

* When two sides are parallel, and the others not, the quadrilateral is sometimes called a trapezoid.

† The circle may be considered as generated by one complete revolution of a radius in one plane; and the circumference as generated by the revolution of the extreme point of the radius.

AXIOMS.

(Assumptions.)

1. Two magnitudes which bear an equal ratio to a third are equal to each other.
2. The relation which two magnitudes bear to each other is not affected by the addition or subtraction of equal magnitudes to or from each of them.
3. Parallel straight lines are equally inclined to any straight line which falls across them.

(These three axioms are the principal *assumptions* made in the propositions which follow. It will be well for the student to compare them with the axioms in Euclid's Elements of Geometry. The first of those above given embraces the first, sixth, and seventh of Euclid. The second embraces the second, third, fourth, and fifth of Euclid. The third is substituted for the twelfth of Euclid. The remaining four of Euclid, which it was not thought necessary to add to these three, are as follows:—

8. Magnitudes which coincide, or which exactly fill the same space, are equal to each other.
9. The whole is greater than its part.
10. Two straight lines cannot enclose a space.
11. All right angles are equal to each other.)

Further Explanation of Terms, &c.

The first stage of Elementary Geometry of two dimensions, treated of in this book, embraces theorems and problems on the properties of straight lines and rectilinear figures, "as far as they are treated of in the First Book of Euclid;" that is to say—

(1.) On the construction and subdivision of straight lines, and of the angles between them;

(2.) On the construction and properties of triangles, and on the equality of certain triangles;

(3.) On the construction and properties of parallel straight lines and parallelograms;

(4.) On a certain property of right-angled triangles, and its converse.

It is not possible to preserve this exact order of classification in the arrangement of the propositions; thus, for instance, we cannot show (theoretically) how to describe a rectilinear angle equal to a given one until we have shown how to describe a triangle equal to a given triangle.

The order in which the propositions come is not precisely the same as that of Euclid, though it is so from Prop. 32 to the end. Whilst, moreover, the subject-matter of the book is necessarily equivalent, yet the methods in some cases are different, and the language is abbreviated.

The only symbol employed is the sign of equality (=); and the student is recommended to use no other, but to write out the whole of the text in full.

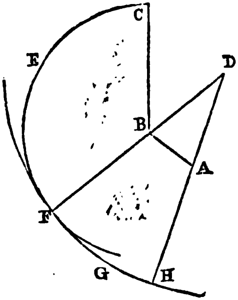
A **Problem** is a proposition of something required to be *done:* as, "to draw a straight line from a given point equal to a given straight line." In order to solve a problem, we must first show how the required task is to be performed, and direct the *construction* of a figure, both for the solution itself, and for the illustration of the proof; and then we must write out a *proof* of the correctness of our solution.

A **Theorem** is a proposition of something required to be *shown*, or *established:* as that "two triangles are equal when two sides of the one are equal to two sides of the other, and when the included angles are equal." In order to establish a theorem, we have generally to provide a *construction* in order to illustrate the proof, and then to write out the *proof* itself.

A **Corollary** is a minor proposition, or deduction, suggested by, and usually established in the course of the principal proposition.

Proposition 1.—Problem.

To draw a straight line from a point (A) *equal to a straight line* (BC).



Construction.—Join AB. With centres A and B, with any equal radiuses AD, BD, draw circles meeting in D. With centre B and radius BC, draw a circle CE. Join DB, and produce it to cut CE in F. With centre D and radius DF, draw a circle FG. Join DA, and produce it to cut FG in H. AH is the line required.

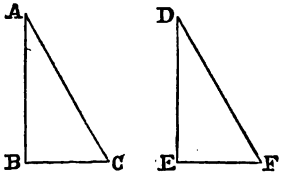
Proof.—The radius DF = the radius DH. But DB, DA were taken equal. Therefore the remainders BF, AH are equal (Ax. 2). Again, radius BF = radius BC. Therefore AH = BF = BC.

Corollary 1.—By the help of this proposition, we can cut off from AB, the greater of two given straight lines, a part equal to C, the less. From A draw AD = C. With centre A, radius AD, draw a circle cutting AB in E. Then AE = AD = C.

Cor. 2.—If the radius AD be taken equal to AB, the triangle ABD will be equilateral.

Proposition 2.—Theorem.

Two triangles are equal when two sides of the one are equal to two sides of the other, and when the included angles are equal.



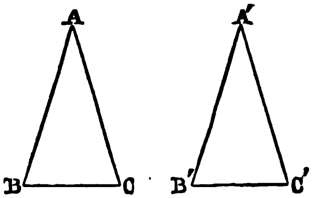
The triangles ABC, DEF, are equal, if AB = DE and AC = DF, and also the angle at A = the angle at D.

Proof.—If A be placed on D, and AB on DE, then B will fall on E, since AB = DE. Also AC will fall on DF, since it has the same inclination to AB as DF has to DE; and C will fall on F, since AC = DF. Again, since B falls on E and C on F, BC and EF are identical.

Hence the *sides* of the triangles, the *angles*, and the *areas* coincide.

Proposition 3.—Theorem.

The angles at the base of an isosceles triangle are equal.

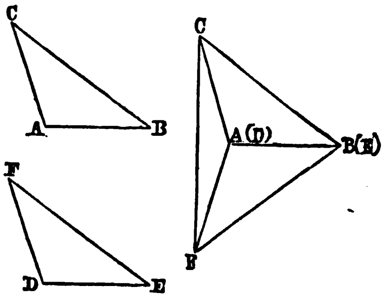


PROOF.—If we take two isosceles triangles, ABC, and A′B′C′, such that AB = A′B′, and AC = A′C′, and also the angle at A = the angle at A′—and, consequently by the last proposition, the angle at B = the angle at B′, and the angle at C = the angle at C′; and if we apply ABC to A′B′C′ reversed, in such manner that AB falls on A′C′, and consequently AC on A′B′; the two triangles will coincide; and the angle at C will coincide with the angle at B′. But the angle at B′ = the angle at B. Therefore the angle at C = the angle at B.

COR.—Hence it follows that an equilateral triangle is also equiangular.

Proposition 4.—Theorem.

Two triangles are equal when the three sides of the one are equal to the three sides of the other.



The triangles ABC, DEF are equal when AB = DE, BC = EF, CA = FD.

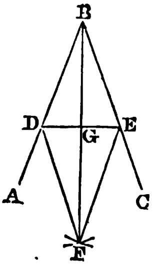
PROOF.—Apply DE to AB, and let the point F fall on the side of AB remote from C. Join CF. Then since BC = BF, the angle BCF = BFC

(Prop. 3). Similarly ACF = AFC. Therefore, by Ax. 2, the remaining angle BCA = the remaining angle BFA.

That is, the original angles C and F are equal, and they are included by equal sides. Therefore the triangles ABC, DEF are equal (Prop. 2).

Proposition 5.—Problem.

To bisect an angle (ABC).

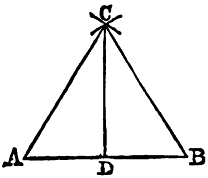


CONSTRUCTION.—Cut off any parts BD, BE, from BA, BC. With centres D and E, with any equal radiuses, draw circles meeting in F. Join BF, DE, intersecting in G. Then BG bisects the angle ABC.

PROOF.—DB, BF, FD = EB, BF, FE respectively; therefore the angle DBF = the angle EBF (Prop. 4).

Proposition 6.—Problem.

To bisect a straight line (AB).

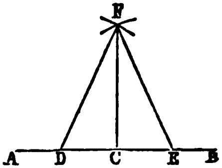


CONSTRUCTION.—With centres A, B, and equal radiuses, draw circles meeting in C. Join AC, CB, and bisect the angle ACB by the straight line CD (Prop. 5). D bisects AB.

PROOF.—AC, CD = BC, CD, and the angles included are equal. Therefore AD = DB (Prop. 2).

Proposition 7.—Problem.

To draw a straight line at right angles to a given straight line (AB), *from a given point in it* (C).

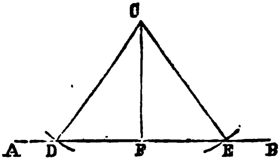


Construction.—In AB take CD and CE equal. With centres D and E, and equal radiuses, draw circles meeting in F. Then FC is at right angles to AB.

Proof.—FC, CD, DF = FC, CE, EF respectively. Therefore the angle FCD = the angle FCE; and since DE is a straight line, the angles at C are right angles (Def. 9).

Proposition 8.—Problem.

To draw a perpendicular to a straight line (AB) *from a point* (C) *outside.*

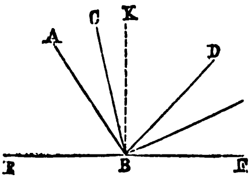


Construction.—With centre C, draw a circle cutting AB in two points, D, E. Join CD, CE, and bisect DE in F. CF is the perpendicular.

Proof.—CF, FD, DC = CF, FE, EC respectively. Therefore angle CFD = angle CFE; and these are right angles (Def. 9).

Proposition 9.—Theorem.

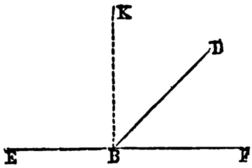
If any number of straight lines (AB, CB, DB, &c.) *cut another straight line* (EF) *in a point* (B), *the angles* (EBA ABC, CBD, *&c.*) *formed on the same side of the last line* (EF) *are together equal to two right angles.*



Draw BK perpendicular to EF (Prop. 7). Then it is manifest that the sum of the inclinations of AB to EB, CB to AB, DB to CB, &c. = the sum of the inclinations of KB to EB and KB to FB. But the inclinations of KB to EB and KB to

FB are two right angles. Therefore the angles EBA, ABC, CBD, &c. = two right angles.

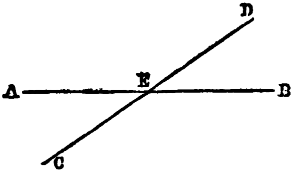
Cor.—From this proposition, or directly from Def. 9, it follows that if two straight lines make with another, upon opposite sides of it, the adjacent angles equal to two right angles, these two straight lines are in one and the same straight line.



Proposition 10.—Theorem.

If two straight lines intersect, the vertical (opposite) angles are equal.

Proof.—Since DE cuts AB in E, the angles DEA, DEB = two right angles. Similarly AEC, AED = two right angles. Therefore DEA, DEB = AEC, AED; and DEB = AEC. For like reasons AED = CEB.

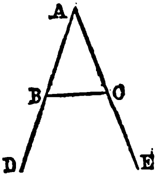


Cor. 1.—From this proposition, or directly from Def. 9, it follows that the angles made by any number of straight lines intersecting at a point are together equal to four right angles.

Proposition 11.—Theorem.

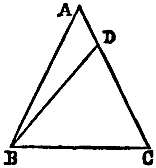
The angles on the outer side of the base of an isosceles triangle, made by producing the equal sides, are equal to each other.

For the two angles at B = two right angles (Prop. 9) = the two angles at C. But ABC = ACB (Prop. 3). Therefore DBC = ECB.



Proposition 12.—Theorem.

If two angles of a triangle are equal, so are the opposite sides.



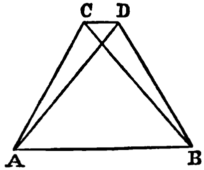
If the angle ABC = the angle ACB, then AC = AB.

PROOF.—Suppose the sides unequal, and AC greater than AB. Cut off CD = BA, and join BD. Then DBC is an isosceles triangle, and DBC = DCB. But we are told that ABC = DCB. Therefore ABC = DBC (Ax. 1), which is absurd. Therefore the supposition is false; and AB = AC.

COR.—Hence it follows that an equiangular triangle is also equilateral.

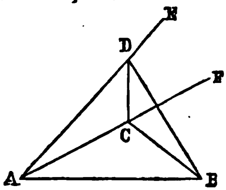
Proposition 13.—Theorem.

On the same base, and on the same side of it, there cannot be two triangles having the sides terminated by one extremity of the base equal, and also the sides terminated by the other extremity of the base equal.



In the triangles CAB, DAB, suppose CA = DA.

Case 1; when C *is outside the triangle* DAB. Join CD. Since AC = AD, the angle ACD = the angle ADC. But ACD is greater than BCD; therefore, also, ADC is greater than BCD. Much more is BDC greater than BCD; therefore the triangle BCD cannot be isosceles. That is, BC cannot be equal to BD.



Case 2; when C *is inside the triangle* DAB. Join CD, and produce AD, AC to E and F. Since ACD is isosceles, EDC = FCD. But EDC is greater than BDC; therefore, also, FCD is greater than BDC. Much more then is BCD

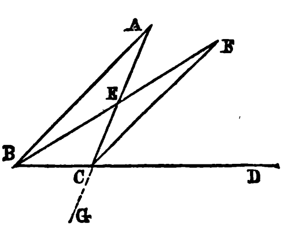
greater than BDC; therefore BC cannot be equal to BD.

Case 3; when C is on DB. Here it is plain that BC cannot be equal to BD.

Proposition 14.—Theorem.

If any side of a triangle be produced, the exterior angle is greater than either of the two interior and opposite angles.

If the side BC of the triangle ABC be produced to D, ACD is greater than BAC or ABC.

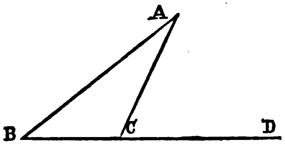


PROOF.—Bisect AC in E; join BE; produce BE, and make EF = BE. Then AE, EB = CE, EF respectively; and the included angles AEB, CEF are equal, being opposite vertical angles (Prop. 10). Therefore the angle BAE = the angle ECF. But ACD is greater than ECF; therefore ACD is greater than BAE. Similarly we may prove that BCG is greater than ABC. But BCG = ACD; therefore ACD is greater than ABC.

Proposition 15.—Theorem.

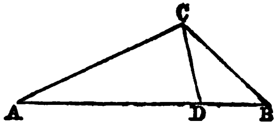
Any two angles of a triangle are less than two right angles.

Produce BC to D. Then ACD is greater than either of the angles at A and B (Prop. 14). Hence ACD and ACB are greater than either of the angles at A and B together with ACB. But ACD and ACB = two right angles. Therefore, the two angles at A and C are less than two right angles; as also the two angles at B and C. Similarly we may show that the third couple of angles, at A and B, are together less than two right angles.



Proposition 16.—Theorem.

If one side of a triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the less.

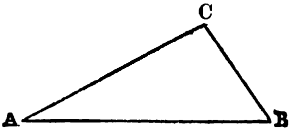


If AB is greater than AC, ACB is greater than ABC.

PROOF.—Cut off AD = AC, and join CD. Then ADC = ACD. But the exterior angle ADC of the triangle CDB is greater than CBD; therefore also ACD is greater than CBD. Much more, then, is ACB greater than CBA.

Proposition 17.—Theorem.

If one angle of a triangle is greater than another, the side opposite the greater angle is greater than the side opposite the less.



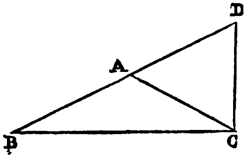
If ACB is greater than ABC, AB is greater than AC.

PROOF.—AB cannot be = AC, for then ABC would be = ACB, which it is not.

AB cannot be less than AC, for then the greater side AC would be subtended by the greater angle ABC, which it is not. Therefore, since AB is neither equal to nor less than AC, AB must be greater than AC.

Proposition 18.—Theorem.

Two sides of a triangle are greater than the third.



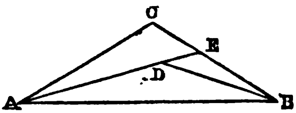
BA and AC are greater than BC.

PROOF.—Produce BA to D, making AD = AC. Join DC. Because AD = AC, ADC = ACD. Therefore BCD is greater than BDC; and consequently BD, that is BA and AC, is greater than BC.

Similarly we may prove AB and BC greater than AC; and BC and CA greater than AB.

Proposition 19.—Theorem.

If from the ends of a side of a triangle (ABC) *two straight lines* (AD, BD) *be drawn to a point within the triangle, these straight lines are less than the other two sides* (AC, BC), *but they contain a greater angle.*

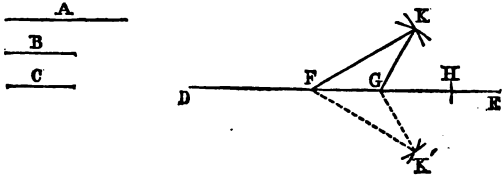


Proof.—Produce AD to meet BC in E. Then, by Prop. 18, AC, CE are greater than AE; and therefore AC, CE, EB, that is AC, CB, are greater than AE, EB. Again, BE, ED are greater than BD; and therefore BE, ED, DA, that is, BE, EA, are greater than BD, DA. But AC, CB are greater than AE, EB. Much greater, then, are AC, CB than AD, DB.

Secondly, by Prop. 14, ADB is greater than DEB, and DEB is greater than ACB. Much greater, then, is ADB than ACB.

Proposition 20.—Problem.

To make a triangle whose sides shall be of given length.



Let A, B, C be the given lengths. (It is necessary that any two of these shall be greater than the third. Prop. 18).

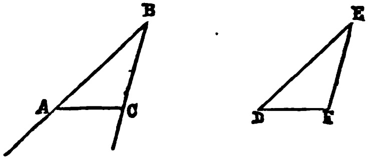
Construction.—Take a straight line DE, and cut off DF = A, FG = B, GH = C. With centre F, radius FD, draw a circle; and with centre G, radius GH, draw

another, cutting the first in K. Join KF, KG. KFG is the triangle required.

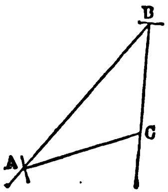
PROOF.—KF = DF = A; KG = GH = C; and FG = B.

Proposition 21.—Problem.

To make a rectilinear angle equal to a given one (ABC).



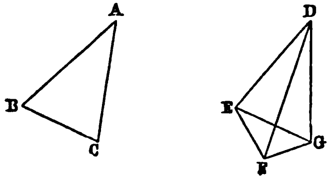
(There are two methods by which this problem can be solved, of which the most general but the least simple is given first). Take any points, A, C, in BA, BC, and join AC. Make the triangle DEF, having DF = AC, FE = CB, ED = BA (Prop. 20). Then DEF = ABC, by Prop. 4.



(2.) Take any point C in BC. With centre C, radius CB, draw a circle cutting BA in A. Then CB = CA, and CAB = CBA (Prop. 3).—The first method, though more complicated in its construction, enables us to draw the angle at *any point* E.

Proposition 22.—Theorem.

If two sides in one triangle are equal to two sides in another, but the included angles are unequal; the base of that which contains the greater angle is greater than the base of the other.



If AB, AC = DE, DF respectively, but BAC greater than EDF; then BC is greater than EF. Assume DE not greater than DF. Make the angle EDG = BAC, and DG = AC = DF. Join EG, GF.

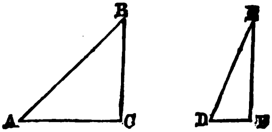
Then since DF = DG, the angle DFG = the angle DGF. But EFG is greater than DFG, therefore greater than DGF, and much more greater than EGF. Consequently EG is greater than EF. Again, since ED, DG = BA, AC, and the included angles are equal, therefore EG = BC (Prop. 2). Wherefore BC is greater than EF.

NOTE.—If DE were greater than DF, the construction and proof would be different.

Proposition 23.—Theorem.

If two sides in one triangle are equal to two sides in another, but the bases are unequal; the angle opposite the greater base is greater than the angle opposite the less.

If AB, BC = DE, EF, but AC is greater than DF, then ABC is greater than DEF. It cannot be equal; for then AC would be equal to DF (Prop. 2), which is not the case. And it cannot be less; for then AC would be less than DF (Prop. 22), which is not the case. Therefore ABC must be greater than DEF.



Proposition 24.—Theorem.

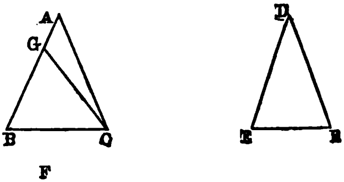
Two triangles are equal when two angles of the one are equal to two angles of the other, and when a side in one is equal to a corresponding side in the other.

The proof differs according as the equal sides are between *(adjacent to)* the equal angles in each, or opposite one of them.

(1.) *When the equal sides are adjacent to both equal angles.*

If ABC, ACB = DEF, DFE respectively, and BC = EF; then AB shall be = DE.

If not, cut off BG = ED, and join CG. Then GB, BC = DE, EF, and

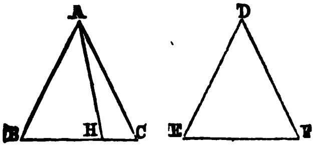


5 E.

GBC = DEF. Therefore the triangles GBC, DEF are equal (Prop. 2), and GCB = DFE = ACB; which is absurd. Hence our supposition is false, and AB = DE. Wherefore AB, BC = DE, EF, and ABC = DEF; and the triangles are equal, by Prop. 2.

(2.) *When the equal sides are opposite corresponding equal angles.*

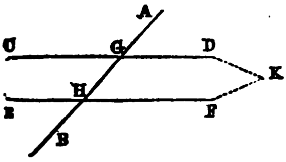
If ABC, ACB = DEF, DFE, and AB = DE; then BC = EF.



If not, cut off BH = EF, and join AH. Then AB, BH = DE, EF, and ABH = DEF; hence the angle AHB = DFE = ACB, which is impossible, by Prop. 14. Therefore our supposition was false; and BC = EF. Wherefore the triangles ABC, DEF are equal, by Prop. 2.

Proposition 25.—Theorem.

If a straight line falling on two others makes the alternate angles equal, these two straight lines are parallel.

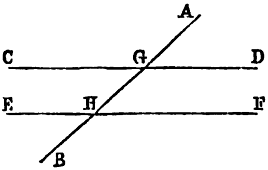


If AB falling on CD and EF make CGH = GHF, CD and EF are parallel.

If not, they meet. Suppose they meet in K. Then GHK is a triangle, and CGH is greater than GHF (Prop. 14). But this is not the case. Therefore CD and EF are parallel.

Proposition 26.—Theorem.

If a straight line falling on two others makes an exterior angle equal to the interior and opposite one on the same side of the single line, the two straight lines are parallel.

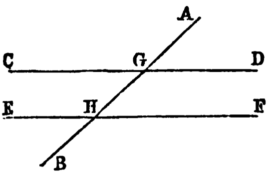


If AB falling on CD and EF make AGD = GHF, CD and EF are parallel.

PROOF. — Since AGD = GHF, and also = CGH (Prop. 10); therefore CGH = GHF; and CD, EF are parallel, by Prop. 25.

Proposition 27.—Theorem.

If a straight line falling on two others makes the two interior angles on the same side of the single line equal to two right angles, the two straight lines are parallel.



If AB falling on CD and EF make CGH, GHE = two right angles; CD and EF are parallel.

PROOF.—Since CGH, GHE = two right angles, and GHE, GHF = two right angles, (Prop. 9); CGH, GHE = GHE, GHF. Hence CGH = GHF; which are alternate. Wherefore CD, EF are parallel.

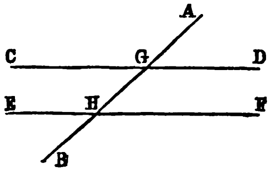
Proposition 28.—Theorem.

If a straight line fall on two parallel straight lines, it makes—

(1.) *The exterior angle equal to the interior and opposite angle on the same side of the line;*

(2.) *The alternate angles equal; and*

(3.) *The two interior angles on the same side of the line equal to two right angles.*



If AB falls on the parallels CD, EF, it makes

(1.) AGD = GHF.

For since CD, EF never meet (Def. 29), they must be similarly inclined to any straight line which crosses them (Ax. 3). Hence AGD = GHF.

(2.) CGH = GHF.

For CGH = AGD = GHF.

(3.) Also DGH and GHF = two right angles.

For AGD = GHF; therefore AGD and DGH = DGH and GHF.

But AGD and DGH = two right angles (Prop. 9).

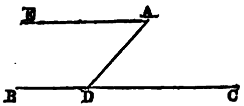
Wherefore DGH and GHF = two right angles.

Note.—This proposition embraces the converse of the last three. It is impossible to prove it without assuming some positive property of parallel straight lines; and the assumption in the text, that parallels have the same inclination to any straight line crossing them, appears one of the most simple. (In writing out the proposition, the student should be careful to draw attention to the nature of his assumption.)

Cor.—It follows from this that straight lines parallel to the same straight line are parallel to each other.

Proposition 29.—Problem.

To draw through a given point (A) *a parallel to a given straight line* (BC).

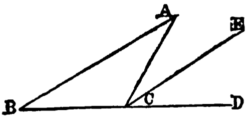


Construction.—Draw AD to any point D in BC. At the point A draw the angle DAE = ADC. Then EA is parallel to BC, by Prop. 25.

Proposition 30.—Theorem.

If one side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles.

If BC be produced to D, the angle ACD = the two angles at A and B.



CONSTRUCTION.—Draw CE parallel to BA.

PROOF.—ACE = the angle at A; and ECD = the angle at B (Prop. 28). Therefore ACD = the angles at A and B.

Proposition 31.—Theorem.

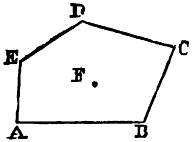
The three angles of a triangle are equal to two right angles.

With the last figure, since ACD = the angles at A and B, add to each side the angle ACB. Then the three angles of the triangle = the angles ACD, ACB = two right angles (Prop. 9).

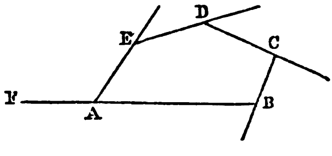
Proposition 32.—Theorem.

The inner angles of a rectilinear figure, with four right angles, are equal to twice as many right angles as the figure has sides.

The figure may be divided into as many triangles as it has sides, by joining the angular points with any point F within the figure. Then (Prop. 31) all the angles of these triangles = twice as many right angles as the figure has sides. But all the angles of these triangles = the inner angles of the figure, with the angles at F; that is, the inner angles of the figure, with four right angles (Prop. 10, *Cor.*) Hence, the inner angles of the figure, with four right angles, are equal to twice as many right angles as the figure has sides.



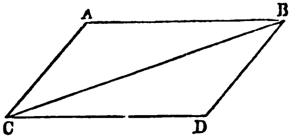
COR.—From this proposition it readily follows that the outer angles of a rectilinear figure, made by producing the sides successively in the same direction, are equal to four right angles.



For all the inner and all the outer = twice as many right angles as the figure has sides (Prop. 9); which are = all the inner, with four right angles (Prop. 23). Therefore all the outer angles = four right angles.

Proposition 33.—Theorem.

The straight lines joining the ends of two equal and parallel straight lines are themselves equal and parallel.

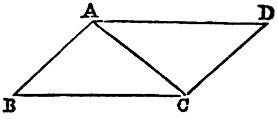


If AB is equal and parallel to CD, AC is equal and parallel to BD.

PROOF.—Join BC. Then AB, BC = DC, CB, and ABC = DCB (Prop. 28); therefore AC = BD, and ACB = DBC (Prop. 2). And since ACB = DBC, AC is parallel to BD (Prop. 25). Hence AC is both equal and parallel to BD.

Proposition 34.—Theorem.

The opposite sides and angles of parallelograms are equal.



Let ABCD be a parallelogram. Join AC.

PROOF.—Since the opposite sides are parallel, the angles DAC, ACB are equal; as also ACD and BAC. And AC, between the equal angles, is common to the two triangles. Therefore these triangles are equal (Prop. 24); and AD = BC, DC = AB. Also

the third angles in the triangles, B and D, are equal; as also are BAD, DCB (Ax. 2).

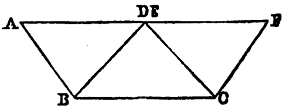
Cor.—It appeared in the proof that the parallelogram was bisected by AC. Hence parallelograms are bisected by their diagonals.

Proposition 35.—Theorem.

Parallelograms on the same base and between the same parallels are equal.

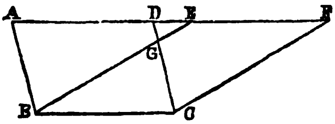
If ABCD, EBCF are on the same base BC, and between the parallels AF, BC; then ABCD = EBCF.

(1.) When D and E coincide.



Since each parallelogram is double of the triangle DBC (Prop. 34, *Cor.*); the parallelograms are equal (Ax. 1).

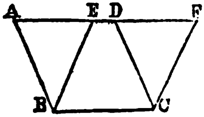
(2.) When AD, EF do not meet.



Let the intersection of BE, DC be G. Since AD = BC = EF; therefore AE = DF (Ax. 2).
Since EA, AB = FD, DC, and BE = CF, therefore the triangle EAB = the triangle FDC (Prop. 4). Take away DGE from both. Therefore the trapezium DABG = the trapezium FEGC.

To these equals add GBC; then ABCD = EBCF.

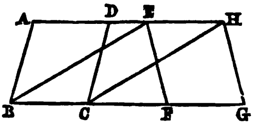
(3.) When AD, EF overlap.



As in case 2, AD = EF, and subtracting ED, AE = DF. Also the triangle EAB = the triangle FDC. Add DEBC to each; then ABCD = EBCF.

Proposition 36.—Theorem.

Parallelograms on equal bases and between the same parallels are equal.



AC, EG, on equal bases BC, FG, and between the parallels BG, AH, are equal.

Join BE, CH. Then BC = FG = EH, and BC, EH are equal and parallel. So also therefore are EB, HC (Prop. 33); and EBCH is a parallelogram.

Hence AC = BH = EG.

Proposition 37.—Theorem.

Triangles on the same base and between the same parallels are equal.

For each triangle is half the parallelogram on the same base and between the same parallels. Therefore the triangles are equal, by Ax. 1.

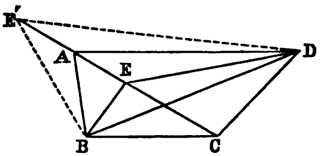
Proposition 38.—Theorem.

Triangles on equal bases and between the same parallels are equal.

For each triangle is half the parallelogram on the same base and between the same parallels; and these parallelograms are equal, by Prop. 36. Therefore the triangles are equal, by Ax. 1.

Proposition 39.—Theorem.

Equal triangles on the same base and on the same side of it are between the same parallels.



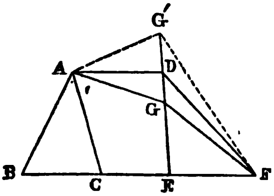
If ABC = DBC, AD is parallel to BC.

If not, let DE be parallel to BC, and let it cut AC, or AC produced, in E. Join EB.

Since DE is parallel to BC, the triangles EBC, DBC are equal (Prop. 37). But so are ABC, BDC. Therefore EBC = ABC; which is absurd. Therefore AD is parallel to BC.

Proposition 40.—Theorem.

Equal triangles on equal bases in the same straight line, both above or both below, are between the same parallels.



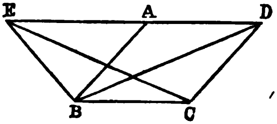
If ABC = DEF, AD is parallel to BF.

If not, let AG be parallel, cutting DE in G. Join GF.

Since AG is parallel to BF, GEF = ABC (Prop. 38). But DEF = ABC. Therefore GEF = DEF; which is absurd. Therefore AD is parallel to BF.

Proposition 41.—Theorem.

A parallelogram is double of all triangles on the same base and between the same parallels.



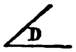
ABCD is double of EBC, where EAD is a straight line. Join BD.

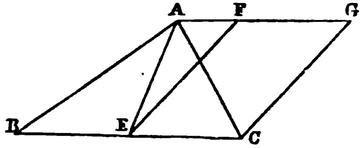
Proof.—ABCD is double of DBC; and EBC = DBC. Hence ABCD is double of EBC.

Proposition 42.—Problem.

To draw a parallelogram equal to a given triangle (ABC), *and having a pair of angles equal to a given angle* (D).

Construction.—Bisect BC in E. At E draw an angle CEF = D (Prop. 21). Through A draw AFG



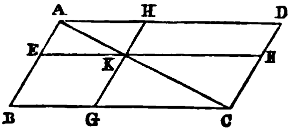


parallel to EC; and through C draw CG parallel to EF. Then FECG is the parallelogram.

Proof.—FECG is a parallelogram, and it is double of AEC (Prop. 41). But ABC is double of AEC (Prop. 38). Therefore FECG = ABC, and has the angles at E and G = D.

Proposition 43.—Theorem.

The complements of the parallelograms formed by drawing parallels to the sides of any parallelogram intersecting in its diagonal are equal.



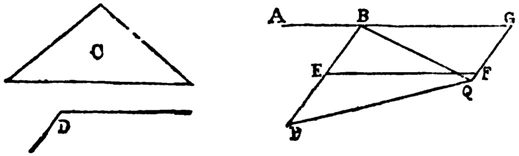
That is, EG = HF.

Proof.—The triangle ABC = the triangle ADC; similarly AEK = AHK, and KGC = KFC (Prop. 34, *Cor*). Therefore the remaining part EG = the remaining part HF.

Proposition 44.—Problem.

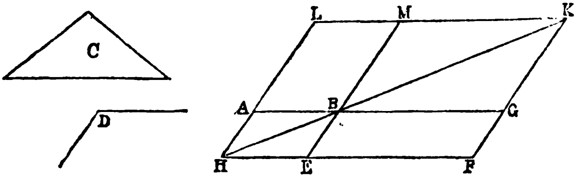
To a given straight line (AB) *to apply a parallelogram equal to a given triangle* (C), *and having a pair of angles equal to a given angle* (D).

CONSTRUCTION.—[(1.) By Prop. 20, taking a straight line BP, terminated in B, and making an angle PBG



with AB produced equal to D, draw a triangle BPQ = C. Bisect BP in E; and (2.)—] *

Make the parallelogram BEFG = C, and having the angle EBG = D.



Complete the parallelogram AE. Join HB, and produce it to meet FG produced in K.

Complete the parallelogram FL. Then BL is the required parallelogram. Produce EB to meet LK in M.

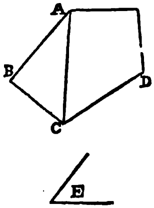
PROOF.—The complement BL = the complement BF = C. And the angle ABM = the vertical angle EBG = D.

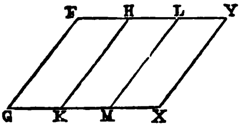
Proposition 45.—Problem.

To draw a parallelogram equal to any rectilinear figure (ABCD...), *and having an angle equal to a given angle* (E).

* It is not necessary to write out the bracketed part of the construction, which is inserted here as an exercise.

Construction.—Draw a parallelogram FGKL equal to the triangle ABC, and having an angle FGK = E.





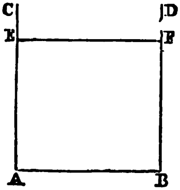
To HK apply a parallelogram HM = ACD, having an angle HKM = E. And so on, for as many triangles as ABCD... consists of. Then FX is the required parallelogram.

Proof.—Since the angle FHK = FGK = E = HKM, therefore FH is parallel to KM. In a similar manner it may be shown that every part of GX is parallel to FH; and that every part of FY is parallel to GK.

Therefore FY and GX are straight lines, and parallels. Hence FX is a parallelogram; and it is equal to the figure ABCD..., and has the angle FGX = E.

Proposition 46.—Problem.

To draw a square on a given straight line (AB).



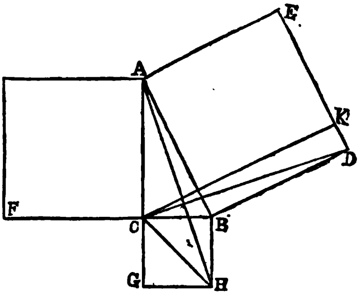
Construction.—Draw AC, BD at right angles. Cut off AE, BF = AB; and join EF. Then AF is a square.

Proof.—AF is equilateral. And since A and B are right angles, AE, BF are parallel (Prop. 27) as well as equal. Therefore AB, EF are parallel (Prop. 33); and AEF, EFB are right angles. Hence AF is a square (Def. 25).

Cor.—Hence it is clear that if one angle of a parallelogram is a right angle, so is each of the others.

Proposition 47.—Theorem.

The square on the hypotenuse of a right-angled triangle is equal to the squares on the sides.



If C is the right angle in ABC, the square on AB = the squares on AC, CB.

Construction.—Draw the squares ABDE, BCGH, AF. Draw CK parallel to BD, and join CD, CH.

Proof.—First, since GCB, ACB are right angles, ACG is a straight line; and it is parallel to BH.

Then, since the triangle CBD and the parallelogram BK are on the same base, and between the same parallels BD, CK, BK is double of CBD.

Again, for a like reason, BG is double of ABH.

Now, since CBH = CBA (being a right angle), therefore the angle ABH = the angle DBC; and the sides AB, BH = DB, BC.

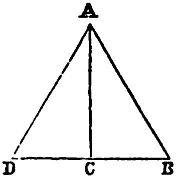
Wherefore the triangle ABH = the triangle DBC; and consequently their doubles GB and BK are equal.

Similarly we may show that FA = AK.

Therefore AK and BK, that is, the square on AB, = the squares on AC and CB.

Proposition 48.—Theorem.

If the square on one side of a triangle is equal to the squares on the other sides, the angle contained by these latter sides is a right angle.



If the square on AB = the squares on AC, CB, ABC is a right angle.

Draw CD at right angles to AC, and make CD = CB. Join AD.

Then ACD is a right-angled triangle, and the square on AD = the squares on AC, CD.

But, since CD = CB, the square on CD = the square on CB. Therefore the squares on AC, CD = the squares on AC, CB.

That is, the square on AD = the square on AB; or AD = AB. Hence, in the triangles ACD, ACB, we have AC, CD = AC, CB, and AD = AB;

Therefore, the angle ACD = the angle ACB.

But ACB is a right angle; therefore, ACB is a right angle.

ALGEBRA.

EXPLANATION OF SYMBOLS.

1. **Algebra** is an abbreviation of Arithmetic, by means of symbols, contracted methods, and the representation of unknown quantities.

2. The letters of the alphabet are used to represent either abstract numbers or concrete quantities. The last letters of the alphabet, especially x, y, and z, usually denote quantities whose numerical value is not known. (Compare the use of x in arithmetical proportion.)

3. The sign of addition is $+$, "plus;" thus, $a + b$. The sign of subtraction is $-$, "minus;" thus, $a - b$. When it is uncertain which of two terms is the greatest, their difference is expressed thus, $a \sim b$. Multiplication is expressed by $\times$, by a point, or by the absence of signs; thus, $a \times b$, $a . b$, or ab. Division is expressed thus, $a \div b$, or $\frac{a}{b}$.

4. The sign of equality is $=$; $\therefore$ signifies "therefore;" $\because$, "because;" $>$, "greater than;" $<$, "less than."

5. When a term is multiplied by itself, it is said to be raised to a **power**. Thus $a \times a =$ second power of a, written a^2; * and inversely, $\sqrt{a^2} = a$. So $a \times a \times a = a^3$; and $\sqrt[3]{a^3} = a$. The square root of a, or $\sqrt{a}$, is also written $a^{\frac{1}{2}}$, or $\frac{1}{a^{-\frac{1}{2}}}$; so $\sqrt[3]{a}$ is written $a^{\frac{1}{3}}$, or $\frac{1}{a^{-\frac{1}{3}}}$. The

* Called "a squared." The powers are also called *dimensions* of the base.

numbers denoting the power of any quantity are called **indices** or **exponents.** The quantities themselves are called the **bases** of the powers. When no index is written, 1 is understood.

6. The *reciprocal* of a is $\frac{1}{a}$ or a^{-1}.

7. A quantity preceded by + is called **positive;** preceded by −, **negative.** When no sign is given, + must be understood. Thus $a - b \equiv + a - b$.

The sign $\equiv$ signifies "is equivalent to."

8. A number multiplying the value of a letter is called a numerical **co-efficient;** as $3\,a$. A letter may be called a literal co-efficient when it precedes another; thus, ax.

9. A **term** is a single quantity or product, as a, ax, $\frac{a}{b}$. An **expression** consists of two or more terms connected by + or −; as $ax + bx - cx$.

10. An expression enclosed in brackets, or under a **vinculum,** is to be treated as a single term. Thus $(ax + bx - cx)$, or $\overline{ax + bx - cx}$. If a co-efficient is common to all the terms in an expression, it may be brought outside a bracket. Thus, $ax + bx - cx = x\,(a + b - c)$, or $x\,.\,\overline{a + b - c}$. A co-efficient outside a bracket is intended to multiply each term inside.

11. The order of the factors in any product is immaterial. Thus, $abcd \equiv bcda$, &c. It is well always to arrange them according to the order of the alphabet, and to place the numerical co-efficients first.

12. When the sum of the indices in each term of an expression is the same, the expression is said to be **homogeneous.** Thus, $a^2b + bcd + c^3$ is homogeneous; as also is $\frac{a^4}{b} + a^2b - \frac{a^4b^4}{c^5}$, or $a^4b^{-1} + a^2b - a^4b^4c^{-5}$.

13. An expression of two terms, as $a + b$, is called a **binomial;** of three terms, a **trinomial;** of more, a **multinomial.**

THE FUNDAMENTAL RULES.

Collection. (Addition and Subtraction.)

14. When we have two or more single terms, in which the *letters* are alike, and also their *signs*, we may add them together by simply adding the co-efficients. Thus, the sum of $3\,ab$ and $5\,ab$ is $8\,ab$; the sum of $-\,5\,ab$ and $-\,4\,ab$ is $-\,9\,ab$; the sum of ax and bx is $(a + b)\,x$.

15. In the same way we may find the difference between two terms of a like nature by subtracting the least from the greatest co-efficient. Thus, the difference of $\pm\,5\,a$ and $\pm\,3\,a$ is $\pm\,2\,a$.

16. But if the *letters* are alike and the *signs* unlike, we must subtract the least from the greatest co-efficient, and the difference will retain the sign of the greatest. Thus, $-\,3\,a + 5\,a = 2\,a$; $3a - 5\,a = -\,2\,a$.

17. If the *letters* are unlike, whatever the signs of the terms may be, we must leave the expression as it is. Thus, $5\,a + 3\,b - 4\,c$ cannot be written in any other form, unless indeed we know the numerical values of a, b, c, and substitute them.

18. And, generally, since the positive or negative signs prefixed to each of the terms in an expression are sufficient in themselves to indicate whether the terms are supposed to be added to or subtracted from those which precede, we need not necessarily employ the terms **Addition** and **Subtraction** in Algebra; but we might use instead the single term **Collection.**

19. When we take b from a, the result is expressed by $a - b$; that is, *a diminished by b.* So $a - (b + c) = a - b - c$.

Again, suppose we have to take $b - c$ from a. It is clear that $a - b$ is less than the true result, for we have taken away b, which is too much by c. We must therefore make the result *greater by a*; that is, $a - (b - c) = a - b + c$.

Or, in another example; suppose we have to subtract $-b$ from a. The term $-b$ denotes, or rather may be conceived as *something less than* 0 *by b*. If, then, we take 0 from a, the result will be less than the true result by b. Hence the true result will be greater than a, by b. Therefore $a - (-b) = a + b$.

20. Generally, therefore, if we have to subtract one expression from another, we must change the signs of all the terms in the subtrahend, and proceed to collect the terms in both minuend and subtrahend. Thus—

$$(a + 2b - 3c) - (2a - 4b + c) = a + 2b - 3c - 2a + 4b - c = -a + 6b - 4c.$$

Multiplication.

21. Since $a^4 \times a^3 = (a \times a \times a \times a), (a \times a \times a) = a^7$ by definition, it is clear that in order to multiply together different powers of the same base, we have simply to add the indices. The numerical co-efficients (if any) are to be multiplied arithmetically; and powers of different bases are to set down side by side. Thus, $3a^2b^3c^4e \times 4a^3bc^2d = 12a^5b^4c^6de$.

When two expressions in brackets are placed side by side, with the sign of multiplication, or without any sign between them, all the terms of one expression are to be multiplied by each term of the other. Thus, $(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$.

22. Now suppose we have to multiply $a - b$ by $c - d$. It is clear that $(a - b)c$ is greater than the true result, because we have multiplied $a - b$ by a term greater than the proper one $(c - d)$, by d. Hence, we must first diminish the result by ad; which gives us $(ac - bc - ad)$. But now the result is too small, for we have taken away ad, when we ought only to have taken away $(a - b)d$. Hence we must compensate the result by *increasing* it by bd. Thus, $(a - b)(c - d) = ac - bc - ad + bd$.

Hence, in multiplying together any two terms, if their

signs are alike (both + or both $-$), the sign of the product must be + ; but, if the signs are unlike, the sign of the product must be $-$.

23. The following examples will show how the multiplication of binomials, trinomials, &c., is to be effected :—

$$\text{(i.)}\quad \begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\quad + ab + b^2} \\ a^2 + 2\,ab + ab^2 \end{array} \qquad \text{(ii.)}\quad \begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad - ab + b^2} \\ a^2 - 2\,ab + b^2 \end{array} \qquad \text{(iii.)}\quad \begin{array}{l} a + b \\ \underline{a - b} \\ a^2 + ab \\ \underline{\quad - ab - b^2} \\ a^2 \qquad\quad - b^2 \end{array}$$

$$\text{(iv.)}\quad \begin{array}{l} a^2 + 2\,ab + b^2 \\ \underline{a + b} \\ a^3 + 2\,a^2b + b^2 \\ \underline{\quad + a^2b + 2\,ab^2 + b^3} \\ a^3 + 3\,a^2b + 3\,ab^2 + b^3 \end{array} \qquad \text{(v.)}\quad \begin{array}{l} a^2 - 2\,ab + b^2 \\ \underline{a - b} \\ a^3 - 2\,a^2b + ab^2 \\ \underline{\quad - a^2b + 2\,ab^2 - b^3} \\ a^3 - 3\,a^2b + 3\,ab^2 - b^3 \end{array}$$

$$\text{(vi.)}\quad \begin{array}{l} a^2 - ab + b^2 \\ \underline{a + b} \\ a^3 - a^2b + ab^2 \\ \underline{\quad + a^2b - ab^2 + b^3} \\ a^3 \qquad\qquad\quad + b^3 \end{array} \qquad \text{(vii.)}\quad \begin{array}{l} a^2 + ab + b^2 \\ \underline{a - b} \\ a^3 + a^2b + ab^2 \\ \underline{\quad - a^2b - ab^2 - b^3} \\ a^3 \qquad\qquad\quad - b^3 \end{array}$$

These seven results are worthy of being remembered. They may be written down in another form :—

(i., ii.).... $(x \pm y)^2 = x^2 \pm 2\,xy + y^2$.
(iii.)....... $(x + y)(x - y) = x^2 - y^2$.
(iv., v.)... $(x \pm y)^3 = x^3 \pm 3\,x^2y + 3\,xy^2 \pm y^3$.
(vi., vii.). $(x \pm y)(x^2 \mp xy + y^2) = x^3 \pm y^3$.

Indices.

24. We have seen that $a^7 \times a^4 = a^{7+4} = a^{11}$; and $a^7 \div a^4 = a^{7-4} = a^3$; or, generally,

$$\left.\begin{array}{l} a^m \times a^n = a^{m+n}; \\ \dfrac{a^m}{a^n} = a^{m-n}; \end{array}\right\} \text{.............(A.)}$$

and hence $\frac{1}{a^n}$ is correctly represented by a^{-n}.

25. Again, if we agree to represent $\sqrt[3]{a}$ by $a^{\frac{1}{3}}$, and $\sqrt[3]{a^2}$ by $a^{2 \times \frac{1}{3}} = a^{\frac{2}{3}}$; or, generally,

$$\sqrt[m]{a} = a^{\frac{1}{m}};$$
$$\sqrt[m]{a^n} = a^{\frac{n}{m}};$$

it is correct to represent $\frac{1}{\sqrt[m]{a}}$ by $a^{-\frac{1}{m}}$ and $\frac{1}{\sqrt[m]{a^n}}$ by $a^{-\frac{n}{m}}$.

Hence we may conclude that the formulas (A) in § 24 are true, whether m and n are integers or fractions, whether positive or negative.

26. What is the value, or meaning, of a^0?

$$\frac{a^m}{a^m} = a^{m-m} = a^0, \text{ whatever } m \text{ may be.}$$

$$\text{But } \frac{a^m}{a^m} = 1 \text{ always.}$$

$$\text{Therefore } a^0 = 1.$$

27. We have assumed that $\sqrt[m]{a^n} = a^{n \times \frac{1}{m}}$. An example will make the truth of the assumption more clear.

$$(a^3)^3 = (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a)$$
$$= a^9, \text{ by definition, } = a^{3 \times 3}.$$

So $\sqrt{a^6} = \sqrt{(a \times a \times a \times a \times a \times a)} = a \times a \times a = a^3 = a^{6 \times \frac{1}{2}}$.

28. It is clear that $\sqrt[m]{a} \times \sqrt[m]{b} = a^{\frac{1}{m}} \times b^{\frac{1}{m}} = (a \times b)^{\frac{1}{m}} = \sqrt[m]{ab}$. Thus, when two factors are under the same index or root, the bases may be multiplied together, and the common root affixed. This operation is sometimes useful in cases like the following:—

Suppose we have to simplify the expression $\frac{\sqrt{\cdot 4}}{\sqrt{10} - \sqrt{\cdot 1}}$.

Neither of these roots can be exhausted precisely, for the

results would be interminable decimals. But, if we multiply numerator and denominator by $\sqrt{10}$, the expression becomes $\dfrac{\sqrt{\cdot 4 \times 10}}{\sqrt{100} - \sqrt{\cdot 1 \times 10}} = \dfrac{\sqrt{4}}{\sqrt{100} - \sqrt{1}} = \dfrac{2}{9}$.

Division.

29. The division of algebraical expressions by binomial or multinomial divisors is analogous to the process of "long division" in arithmetic. It is necessary, however, in the first place, to arrange both dividend and divisor according to ascending or descending powers of some "letter of reference" occurring at least once in each expression.

30. The following examples will illustrate the methods of division :—

$$\frac{12\,a^3b^2 - 15\,a^4bc^3 + 24\,ab^2c - 3\,abc}{3\,abc} = 4\,a^2bc - 5\,a^3c^2 + 8\,b - 1.$$

$$\begin{array}{l} x - y)x^2 - 2\,xy + y^2(x - y. \\ \quad\;\; \underline{x^2 - xy} \\ \qquad\quad - xy + y^2 \\ \qquad\quad \underline{- xy + y^2} \end{array} \qquad \begin{array}{l} x - y)x^3 - y^3(x^2 + xy + y^2 \\ \quad\;\; \underline{x^3 - x^2y} \\ \qquad\; + x^2y \\ \qquad\; \underline{x^2y - xy^2} \\ \qquad\qquad + xy^2 - y^3 \\ \qquad\qquad \underline{xy^2 - y^3} \end{array}$$

With the last two examples compare (ii.) and (vii.) of § 23.

$$\begin{array}{l} a^{\frac{1}{2}} - b^{\frac{1}{2}})a^2 - 2\,ab + b^2(a^{\frac{3}{2}} + ab^{\frac{1}{2}} - a^{\frac{1}{2}}a - b^{\frac{3}{2}}. \\ \qquad\quad \underline{a^2 - a^{\frac{3}{2}}b^{\frac{1}{2}}} \\ \qquad\qquad\quad a^{\frac{3}{2}}b^{\frac{1}{2}} - 2\,ab \\ \qquad\qquad\quad \underline{a^{\frac{3}{2}}b^{\frac{1}{2}} - ab} \\ \qquad\qquad\qquad\qquad - ab \\ \qquad\qquad\qquad\qquad \underline{- ab + a^{\frac{1}{2}}b^{\frac{3}{2}}} \\ \qquad\qquad\qquad\qquad\qquad - a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2 \\ \qquad\qquad\qquad\qquad\qquad \underline{- a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2} \end{array}$$

Common Measures and Multiples.

31. The remarks contained in § 33—37 of the Arithmetic, with respect to measures and multiples of numbers, are equally applicable to algebraical expressions. The process for finding the G.C.M. and the L.C.M. of two or more algebraical quantities is precisely the same as for arithmetical quantities.

32. In finding the *greatest common measure* of two quantities, the dividend or divisor at any stage may be multiplied or divided by any term or expression. The multiplication is useful to avoid the introduction of fractional co-efficients; the division (when there happen to be factors common to every term) abbreviates the work.

Find the G.C.M. of $2a^4 + 4a^2b + 2b^2 - 2c^2$;

and $a^4 + 2a^2b + 2a^2c + b^2 + 2bc + c^2$.

(Remove the factor 2 from the first quantity.)

$$
\begin{array}{r|l}
a^4 + 2a^2b + b^2 - c^2) & a^4 + 2a^2b + 2a^2c + b^2 + 2bc + c^2 (1 \\
 & \underline{a^4 + 2a^2b \qquad\quad + b^2 \qquad\quad - c^2} \\
 & \qquad\qquad 2a^2c \qquad + 2bc + 2c^2
\end{array}
$$

(Remove the factor $2c$.)

$$
\begin{array}{r|l}
a^2 + b + c) & a^4 + 2a^2b + b^2 - c^2 (a^2 + b - c \\
 & \underline{a^4 + a^2b + a^2c} \\
 & \quad a^2b - a^2c + b^2 - c^2 \\
 & \quad \underline{a^2b + b^2 + bc} \\
 & \qquad - a^2c - bc - c^2 \\
 & \qquad \underline{- a^2c - bc - c^2}
\end{array}
$$

The G.C.M. $= a^2 + b + c$.

33. The least common multiple of two or more quantities, when it cannot be found by inspection, must be got by either of the methods employed in arithmetic. Thus—

L.C.M. of $a^2 + 2a - 3$ and $a^2 + 5a + 6$.

$$\text{G.C.M.} = a + 3, \therefore \text{L.C.M.} = \frac{a^2 + 2a - 3}{a + 3}$$

$$\times (a^2 + 5a + 6) = (a - 1)(a^2 + 5a + 6)$$
$$= a^3 + 4a^2 + a - 6.$$

L.C.M. of $x^2 - 2xy + y^2, x^2 - y^2, x^3 - y^3, x^4 - y^4$.

$$\begin{aligned}
x^2 - 2xy + y^2 &= (x - y)(x - y).\\
x^2 - y^2 &= (x - y)(x + y).\\
x^3 - y^3 &= (x - y)(x^2 + xy + y^2).\\
x^4 - y^4 &= (x^2 + y^2)(x + y)(x - y).\\
\therefore \text{L.C.M.} &= (x - y)^2 (x + y)(x^2 + y^2)(x^2 + xy + y^2).
\end{aligned}$$

34. Examples.

I. (§ 1—13)—When $a = 0, b = 1, c = 2, d = 3, e = 4, f = -5$; find the arithmetical values of the following expressions:—

$a + 2b - 3c + 4d + 2f$.

$2ab + 2bc + 3cd + 2ef$.

$a(b + c) + b(c + d)$.

$a^2 + b^2 + c^2 - f^2 + bcd$.

$\sqrt{2bc} + \sqrt{12de} - \sqrt{b - 3f}$.

$\dfrac{\sqrt{abc} + \sqrt[3]{cbe}}{b^{27} + e^{\frac{1}{2}}}$.

$2(a + b) - 2(b + c) - 2(c + d)$.

$5abcd + 5bcde + cdef$.

$ab(b + c + d) - bd(e + f) - ce(b - 2f)$.

$\dfrac{ab + cd}{e} - \dfrac{cd + de}{f}$.

$\sqrt[3]{9d} + 2\sqrt[4]{2bce} - 7\sqrt[5]{b^3}$.

$\dfrac{(\cdot 5d + \cdot 25e)(a + c - b)}{f}$.

II. Add the following expressions, two or more, together; and subtract any one from any other:—

$2a + 3b + 4c - e$.

$2b + 3c - 4e + a$.

$3d + 5c + 4a - b$.

$c + 2e - 5a + b$.

$9b + 5a - 6c + 5e$.

$2abc - 3cde + 4bcd - 5def$.

$6def - 7bcd + 4cde + 11abc$.

$8cde - 11abc - 4bcd + def$.

$6abc + 8dfe - 5cdb + cde$.

$cba - efd + 10bdc - 5ced$.

III. Multiply—

$a^3 + 3a^2b + 3ab^2 + b^2$ by $a + b$, $a + 2b$, $a - 3b$.

$a^4 + 5a^2b^2 - 4a^3b - 2ab^3 + b^4$ by $a - b$, $a^2 - ab + b^2$.

$a^4 + a^3b + a^2b^2 + ab^3 + b^4$ by $a - b$, $a^2 - b^2$, $a - 2b$.

$a^{\frac{1}{2}}b^{\frac{1}{2}} + ab + a^{\frac{3}{2}}b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

IV. Divide—

$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a - b$, $a^2 - 2ab + b^2$, $a^3 - 3a^2b + 3ab^2 - b^3$.

$a^5 - 4a^2b + 6a^3b^2 - 5a^2b^3 + 3ab^4 - b^5$ by $a^3 - 2a^2b + ab^2 - b^3$, $a - b$.

$a^3 + 3ab^2 - b^3 - 3a^2b$ by $a - b$, $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

V. Find the G.C.M. of $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 + a^2b + ab^2 + b^3$; of $a^6 - x^6$ and $a^3 - 3ax(a - x) - x^3$; of $a^2 + b^2 + x^2 - 2a(b + x) + 2bx$ and $a^2 - 2ab + b^2 - x^2$.

VI. Find the L.C.M. of $a + x$, $a - x$, $a^2 + x^2$, $a^4 - x^4$; of $(a^2 + xy)^2$, $(a^2 - xy)^2$, $a^8 - x^4y^4$; of $a^3 + a^2x + ax^2 + x^3$, $a - x$, $a^3 + x^3$.

Fractions.

35. The rules for algebraical fractions are exactly analogous to the rules for arithmetical ratio-fractions. The following examples will illustrate the mode of working:—

$$\frac{x + y}{x - y} + \frac{x - y}{x + y} = \frac{(x + y)^2 + (x - y)^2}{(x - y)(x + y)} = \frac{2(x^2 + y^2)}{x^2 - y^2}.$$

$$\frac{x + y}{x - y} - \frac{x - y}{x + y} = \frac{(x + y)^2 - (x - y)^2}{(x - y)(x + y)} = \frac{4xy}{x^2 - y^2}.$$

$$\frac{3x}{x^2 - y^2} \times \frac{x + y}{3} \times \frac{x - y}{x} = 1.$$

$$\frac{x - 1}{x - 4} \div \frac{x^2 - 1}{x^2 - 16} = \frac{x - 1}{x - 4} \times \frac{x^2 - 16}{x^2 - 1} = \frac{x + 4}{x + 1}.$$

36. The expression $a + x + \frac{x^2}{a - x}$, is *in the form of* a mixed number, and may be treated like $2\frac{3}{4}$ $\left(= \frac{2 \times 4 + 3}{4} = \frac{11}{4}\right)$.

Thus, $a + x + \dfrac{x^2}{a - x} = \dfrac{(a^2 - x^2) + x^2}{a - x} = \dfrac{a^2}{a - x}$.

37. To reduce an algebraical fraction to its lowest terms—in other words, to *simplify* it, we must, as with

arithmetical ratio-fractions, divide numerator and denominator by their G.C.M. This can generally be effected by inspection; namely, by resolving numerator and denominator into their elementary factors, and striking the common factors out of both.

38. One of the greatest difficulties of algebra is overcome when the student can resolve expressions into their elementary factors, with ease and rapidity. Familiarity with such formulas as those given in § 23, greatly assists this process. We add a few more useful hints.

39. Since
$$\begin{aligned}(x+a)(x+b) &= x^2+(a+b)x+ab,\\ (x+a)(x-b) &= x^2+(a-b)x-ab,\\ (x-a)(x-b) &= x^2-(a+b)x+ab,\end{aligned}$$
it is clear that we may resolve any expression of the form $x^2 \pm px \pm q$ into its two elementary factors, or into two factors further resolvable, by finding two numbers or letters (a, b) such that q is their product, and p their algebraical sum.

Thus,
$$\begin{aligned}a^2+5a+6 &= (a+3)(a+2).\\ a^2+a-6 &= (a+3)(a-2).\\ a^2-5a+6 &= (a-3)(a-2).\\ x^2-x-72 &= (x-9)(x+8).\\ x^2+x-72 &= (x+9)(x-8).\\ x^2+17x+72 &= (x+9)(x+8).\\ x^2+2ax+a^2-b^2 &= x^2+(\overline{a+b}+\overline{a-b})x\\ &\quad +(a+b)(a-b)\\ &= (x+a+b)(x+a-b).\\ x^2+2bx-a^2+b^2 &= x^2+(\overline{a+b}-\overline{a-b})x\\ &\quad -(a+b)(a-b)\\ &= (x-a+b)(x+a+b).\end{aligned}$$

The last two cases might have been more simply resolved by observing that
$$\begin{aligned}x^2+2ax+a^2-b^2 &= (x+a)^2-b^2,\\ x^2+2bx-a^2+b^2 &= (x+b)^2-a^2,\end{aligned}$$
and referring them to § 23, iii.

40. It will be found that

$x^n - a^n$ is divisible without remainder
by $x + a$ when n is *even;* and
by $x - a$ whether n is *even* or *odd.*
$x^n + a^n$ „ by $x + a$ when n is *odd,* but
by $x - a$ never.

These facts may be remembered by observing that

$\dfrac{x^2 - a^2}{x \pm a}$ leaves no remainder;

$\dfrac{x - a}{x - a}$ „ „

$\dfrac{x + a}{x + a}$ „ „

The results of such divisions may always be written down after the following patterns:—

$$\frac{x^8 - a^8}{x \pm a} = x^7 \pm x^6 a + x^5a^2 \pm x^4a^3 + x^3a^4 \pm x^2a^5 + xa^6 \pm a^7.$$

$$\frac{x^7 - a^7}{x - a} = x^6 + x^5a + x^4a^2 + x^3a^3 + x^2a^4 + xa^5 + a^6.$$

$$\frac{x^7 + a^7}{x + a} = x^6 - x^5a + x^4a^2 - x^3a^3 + x^2 a^4 - xa^5 + a^6.$$

Surds.

41. An expression indicating the extraction of a root which cannot be found in precise terms is called a **surd**. Thus, $\sqrt{2}$, $\sqrt[3]{4}$ are surds. It is sometimes possible to simplify quantities involving surds, without having to perform the extraction indicated. One example has already been given in § 28.

When the expression is in the form of a fraction, we can generally *simplify the denominator* by making use of the formula $(a + b) \times (a - b) = a^2 - b^2$.

Thus, $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$, where we may take $\sqrt{a}$ and $\sqrt{x}$, to represent **quadratic surds.**

Multiply numerator and denominator by $\sqrt{a} + \sqrt{x}$; then

$$\frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}} = \frac{(\sqrt{a} + \sqrt{x})^2}{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})} = \frac{a + x + 2\sqrt{ax}}{a - x}.$$

Again, to take an arithmetical example—

$$\frac{\sqrt{12} - \sqrt{3}}{\sqrt{12} + \sqrt{3}} = \frac{(\sqrt{12} - \sqrt{3})^2}{(\sqrt{12} + \sqrt{3})(\sqrt{12} - \sqrt{3})} = \frac{12 + 3 - 2\sqrt{36}}{12 - 3}$$

$$= \frac{15 - 12}{9} = \frac{3}{9} = \frac{1}{3}.$$

The formulas in § 40 show that a factor may be found to rationalize any binomial surd. Thus—

$$x^{\frac{1}{2}} + y^{\frac{1}{3}} = \frac{x^3 - y^2}{x^{\frac{5}{2}} - x^{\frac{4}{2}}y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - x^{\frac{2}{2}}y^{\frac{3}{3}} + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{2}}},$$ as will

be found on trial. The binomial $x^3 - y^2$ may be written $x^{\frac{6}{2}} - y^{\frac{6}{3}}$, or $(x^{\frac{1}{2}})^6 - (y^{\frac{1}{3}})^6$. Thus, the rationalizing factor of $x^{\frac{1}{2}} + y^{\frac{1}{3}} = x^{\frac{5}{2}} - x^2y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - xy + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{2}}$. And similarly for $x^{\frac{1}{2}} - y^{\frac{1}{3}}$.

42. Examples.

I. Perform the operations indicated in each case.

(1.) $\dfrac{a + x}{a - x} + \dfrac{a - x}{a + x} - \dfrac{a + x}{a^4 - x^4}$.

(2.) $\dfrac{x - y^2}{x^2 - y} \times \dfrac{x^4 - 2x^2y + y^2}{x^3 - y^6} \times \dfrac{x^2 + xy^2 + y^4}{x^2 + y} \times (x^4 - y^2)$.

(3.) $\dfrac{1}{x-y-z}+\dfrac{(x-y)^3-z^3-3\,(x-y-z)\,(xz-yz)}{x^2+y^2+z^2-2\,(xy+xz+yz)}.$

(4.) $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{2\,a}\times\dfrac{a}{a^2+x^2}$

(5.) $a+x\left\{\dfrac{1}{a-x}+\dfrac{\dfrac{a+x}{a-x}}{\dfrac{a^3+x^3}{a^3-x^3}}\times\dfrac{a^2-ax+x^2}{a^2+ax+x^2}\right\}\dfrac{a-x}{a-x+1}.$

II. Resolve into elementary factors—

(1.) $x^2-7\,x+12.$ (2.) $x^8-2\,x^4-35.$
(3.) $x^2+4\,xy^2-32\,y^4.$ (4.) $a^2b^2-17\,abx-60^2x^2.$

III. Write down the quotients of—

(1.) $\dfrac{x^8-y^8}{x-y}.$ (2.) $\dfrac{a^{10}+b^{10}}{a^2+b^2}.$ (3.) $\dfrac{x^9-y^9}{x-y}.$

(4.) $\dfrac{(a+x)^2+(b+y)^2+2\,(ab+ay+bx+by)}{a+b+x+y}.$

IV. Simplify the expressions—

(1.) $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}.$ (2.) $\dfrac{\sqrt[3]{17\cdot28}-\sqrt[3]{\cdot01}}{\sqrt[3]{3\cdot43}}.$

And rationalize the denominators of—

(3.) $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{20}}.$ (4.) $\dfrac{12-\sqrt{3}}{12+\sqrt{3}}.$ (5.) $\dfrac{a^{\frac{1}{2}}-b^{\frac{1}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{2}}}.$

V. Simplify the following expressions :—

(1.) $\dfrac{1}{x-y}+\dfrac{1}{x+y}-\dfrac{1}{x^2-y^2}+\dfrac{x^3+y^3}{x+y}.$

(2.) $\left(\dfrac{1}{a+b+c}+\dfrac{1}{a-b+c}+\dfrac{1}{b+c-a}\right).\,(a^2+c^2$
$+\,2\,ac-b^2)\,(c-a+b).$

(3.) $\dfrac{1}{a+b} \times \dfrac{1}{a+c} \times \dfrac{1}{a+d} \times \dfrac{a^4 + b^2d^2 - a^2b^2 - a^2d^2}{(a-b)(a-d)}.$

(4.) $\dfrac{\dfrac{1}{l-m} - \dfrac{1}{m-n}}{\dfrac{1}{l+m} + \dfrac{1}{m+n}} \cdot (lm + ln + mn + m^2).$

(5.) $\dfrac{1}{a + \dfrac{1}{a + \dfrac{1}{a + \dfrac{1}{a+1}}}}.$

(6.) $\dfrac{x^2y^2 - 53\,xyz - 54\,z^2}{xy + z}.$

(7.) $\dfrac{15\,x^2 - 2\,xy - 24\,y^2}{3\,x - 4\,y}.$

(8.) $\frac{1}{3}$ of $\dfrac{84\,l^2 - 124\,lmn + 32\,m^2n^2}{3\,l - mn}.$

(9.) $\dfrac{\sqrt{\cdot 000064} - \frac{1}{1000000}}{199999}.$

(10.) $\dfrac{5^{\frac{1}{2}}}{8 - 5\sqrt{5}}.$

Simple Equations.

43. When two algebraical expressions, one or both of which involve a single unknown quantity, are equal to each other, the statement of the equality is called an **equation.** The value of the unknown quantity may be found from an equation in terms of the known quantities; and this process is called the **solution** of the equation. An equation is called **simple** when only the first powers of the unknown quantities enter into it; **quadratic** when the second powers are involved.

44. The solution of an equation proceeds upon the axiom that, if the two sides or members of it be affected in a precisely similar manner, the equality is not destroyed. Thus we may add a quantity to, or subtract it from, one side; or we may multiply or divide one side by any quantity, if we perform the same operation with the other side. For instance, we may transpose any term from one side to the other, changing + to −, or − to +, since this merely amounts to subtracting the said term from *both sides*.

45. The general method of solving simple equations may be thus stated. Remove ratio-fractions by multiplying throughout by the L.C.M. of the denominators. Bring the terms involving the unknown to the left side; the others to the right. Divide both sides by the coefficient of the unknown.

Thus, solve $\frac{x+1}{3} - \frac{x+3}{5} = 2x + \frac{x-81}{2}.$

Multiply throughout by 30:

$$10(x+1) - 6(x+3) = 60x + 15(x-81);$$

or, $$10x + 10 - 6x - 18 = 60x + 15x - 1215.$$

Collect the terms:

$$10x - 6x - 60x - 15x = -1215 - 10 + 18;$$

or, $$-71x = -1207.$$

Multiply by − 1:

$$71x = 1207.$$

Divide by the co-efficient of x:

$$x = \frac{1207}{71} = 17.$$

46. Some equations appear to be quadratics, when they are really simple. Thus;

$$\frac{x+1}{3x+2} = \frac{4x+5}{12x+1}.$$

$$\therefore\ (x+1)(12x+1) = (4x+5)(3x+2);$$

or, $$12x^2 + 13x + 1 = 12x^2 + 23x + 10;$$

or, $$13x + 1 = 23x + 10.$$

$$\therefore\ 10x = -9$$

$$x = \frac{-9}{10}.$$

Again, $\sqrt{1+x} - \sqrt{1-x} = \sqrt{2}.$

$\therefore\ 1 + x + 1 - x - 2\sqrt{1-x^2} = 2$, by squaring.

$$\therefore\ -\sqrt{1-x^2} = 0;$$

or, $$1 - x^2 = 0,$$

$$x^2 = 1,$$

$$x = \pm 1.$$

47. We add a few examples of simple equations, worked out for the assistance of the student.

(1.) $$\frac{1}{a-x} + \frac{1}{a+x} = \frac{x}{a^2-x^2}.$$

Since $$\frac{a+x+a-x}{(a-x)(a+x)} = \frac{x}{a^2-x^2},$$

$$\frac{2a}{a^2-x^2} = \frac{x}{a^2-x^2};$$

$$\therefore\ x = 2a.$$

(2.) $$\frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2+5x+6}.$$

Here it is evident that $(x+2)(x+3) = x^2 + 5x + 6$. Multiplying by the L.C.M. of the denominators—

$$4(x+3) + 7(x+2) = 37,$$

$$11x + 26 = 37,$$

$$11x = 11,$$

and $x = 1$.

Since A can do the work in 9 days, he can do $\frac{1}{9}$ of the work in 1 day.

Similarly B can do $\frac{1}{12}$ of the work in 1 day.

Let x = the number of days required.

Then A does $\frac{x}{9}$ of the work,

and B does $\frac{x}{12}$ of the work.

Now $\frac{x}{9}$ of the work + $\frac{x}{12}$ of the work = the whole work.

Hence we conclude that $\frac{x}{9} + \frac{x}{12} = 1,$

$$\begin{aligned} \text{or } 12\,x + 9\,x &= 84, \\ 21\,x &= 84, \\ x &= 4 \text{ days.*} \end{aligned}$$

Again: A and B lay a wager of 10s.; if A loses he will have 25s. less than twice as much as B will then have; but if B loses he will have $\frac{5}{17}$ of what A will then have. How much money had each?

Let x = what A has.

Then, if A loses, he will have x – 10s., which is 25s. less than twice what B now has. Therefore—

$$\begin{aligned} x - 10 + 25 &= \text{twice what B now has.} \\ \therefore \tfrac{1}{2}(x + 15) &= \text{what B now has.} \\ \therefore \tfrac{1}{2}(x + 15) - 10 &= \text{what B had at first, since B} \end{aligned}$$

won the bet.

* It will be useful to compare the arithmetical solution of the question.

Since A does the work in 9 days, he does $\frac{1}{9}$ in one day.
Since B does the work in 12 days, he does $\frac{1}{12}$ in 1 day.
Therefore together they do $\frac{1}{9} + \frac{1}{12} = \frac{21}{84}$, in 1 day.

Hence they can do the whole in $\frac{1}{\frac{21}{84}} = \frac{84}{21} = 4$ days.

In both processes we take the same step, passing from the concrete term, "the whole work," to the abstract term, *unity.*

Now, consider the second condition of the question, when A wins. Thus—

$$\frac{1}{2}(x + 15) - 10 - 10 = \frac{5}{17}(x + 10).$$

This is the simple equation resulting from the problem. Multiply each term by 34. Then—

$$\begin{aligned} 17(x + 15) - 680 &= 10(x + 10), \\ \text{or} \quad 17x - 10x &= 100 + 680 - 255, \\ \text{or} \quad 7x &= 525, \\ \text{and} \quad x &= 75 \text{ shillings.} \end{aligned}$$

Thus, A at first had 75 shillings, and B had $\frac{1}{2}(x + 15) - 10 = 35$ shillings.

Simple Equations involving more than one Unknown Quantity.

50. Here we must have as many separate and distinct equations as there are unknown quantities. Such equations are called *simultaneous*.

I. For Two Unknown Quantities.

There are three modes of solution.

(1.) Multiply one or both equations throughout, so as to make the co-efficients of one unknown the same in both. Add or subtract corresponding sides of the equations, according as the signs are different or alike; which will cause one unknown to disappear. The resulting equation will only involve the other unknown. Solve this by the preceding rules; and, to obtain the value of the other unknown, substitute the value just found in either of the original equations.

$$\text{Thus:} \quad \left.\begin{aligned} 2x + 3y &= 7 \ldots \text{i.} \\ x - 2y &= 0 \ldots \text{ii.} \end{aligned}\right\}$$

Multiply (i.) by 2, and (ii.) by 3.

$$\therefore \quad \left.\begin{aligned} 4x + 6y &= 14 \\ 3x - 6y &= 0 \end{aligned}\right\}.$$

Add; then

$$7x = 14,$$
$$x = 2.$$

Now (i.) becomes $4 + 3y = 7$, and $y = 1$.

(2.) Find the value of one unknown in terms of the other, from each equation separately; and equate these values.

Thus: from (i.) $x = \dfrac{7 - 3y}{2}$,

from (ii.) $x = 2y$.

$$\therefore \quad \frac{7 - 3y}{2} = 2y,$$
$$7 - 3y = 4y,$$
$$\therefore \quad y = 1; \text{ and } x = 2y = 2.$$

(3.) Find the value of one unknown in terms of the other from one of the equations, and substitute this value in the other.

Thus: from (i.) $x = \dfrac{7 - 3y}{2}$,

$$\therefore \text{ (ii.) becomes } \frac{7 - 3y}{2} - 2y = 0,$$
$$\therefore \quad 7 - 3y - 4y = 0,$$
$$\therefore \quad y = 1; \text{ and } x = \frac{7 - 3y}{2} = \frac{4}{2} = 2.$$

II.—For Three or more Unknown Quantities.

51. Apply any of the processes just described to the equations, taken two at a time; aiming at the elimination of two or more unknowns, so as to get a simple equation of one unknown.

Example.—
$$\left.\begin{aligned} 2x + 3y + 4z &= 58 \ldots \text{ i.} \\ 3x + 2y - z &= 16 \ldots \text{ ii.} \\ 4x - 2y + 2z &= 20 \ldots \text{iii.} \end{aligned}\right\}$$

Eliminate z from (i.) and (ii.); and from (ii.) and (iii.):

$$\left.\begin{aligned} 14x + 11y &= 122 \\ 10x + 2y &= 52 \end{aligned}\right\} \text{...............(A.)}$$

By multiplication these become:

$$\left.\begin{aligned} 28x + 22y &= 244 \\ 110x + 22y &= 572 \end{aligned}\right\}$$

By subtraction:

$$82x = 328$$
$$\therefore \quad x = 4.$$

From (A), the simplest of the secondary equations, we find $y = 6$; and from (ii.), $z = 8$.

Examples of Problems producing Simultaneous Equations.

52. There are two numbers whose sum is a, and their difference is two-thirds of the greater. Find them.

Let x = the greater,
and y = the less.

$$\left.\begin{aligned} \text{Then } x + y &= a \\ \text{and } \quad x - y &= \frac{2x}{3} \end{aligned}\right\}$$

$$\therefore \quad 2x = a + \frac{2x}{3}, \text{ (by addition,)}$$

$$\frac{4x}{3} = a,$$

$$x = \frac{3a}{4}.$$

$$\text{Also, } y = a - x = \frac{a}{4}.$$

Again: A certain number of two digits contains the sum of its digits four times, and their product twice. What is the number?

$$\text{Let } x = \text{the units' figure,}$$
$$y = \text{the tens' figure.}$$

$$\text{Then } \left.\begin{aligned} 10\,y + x &= 4\,(x + y) \\ 10\,y + x &= 2\,xy \end{aligned}\right\}\ldots\ldots\ldots(\text{A.})$$

$$\therefore\ 4\,(x + y) = 2\,xy,$$
$$\therefore\ 2\,x - xy = -\,2\,y,$$
$$x\,(2 - y) = -\,2\,y,$$
$$x = \frac{-\,2\,y}{2 - y} = \frac{2\,y}{y - 2}.$$

Substituting this value of x in the second equation of (A), we have

$$10\,y + \frac{2\,y}{y - 2} = \frac{4\,y^2}{y - 2},$$

$$\text{or, } 5 + \frac{1}{y - 2} = \frac{4\,y}{y - 2},$$

$$\text{or, } 5\,(y - 2) + 1 = 2\,y.$$
$$\therefore\ 3\,y = 9,$$
$$y = 3,$$

$$\text{and } x = \frac{2\,y}{y - 2} = \frac{6}{1} = 6.$$

Therefore the number is 36.

53. Examples on Simple Equations.

I. Solve the equations :

(1.) $2\,x + 7 = 17.$ (2.) $3\,x - 5 = x + 1.$

(3.) $23 - x = 6\,x + 2.$ (4.) $(x + 1)\,(x - 3) = x^2 - 13.$

(5.) $\dfrac{x - 3}{x - 4} = \dfrac{x}{x + 14}.$ (6.) $(x + 2)^2 + (x - 3)^2 = 2\,(x^2 + 3).$

(7.) $(x + 9)\,(2\,x - 3) = 2\,(x + 1)\,(x - 1).$

(8.) $\dfrac{x^3 + 27}{3\,x + 1} = \dfrac{x + 3}{3}.$

(9.) $\left.\begin{array}{l} x+y=5 \\ x-y=1 \end{array}\right\}.$ (10.) $\left.\begin{array}{l} 2x+3y=22 \\ 5x-4y=9 \end{array}\right\}.$

(11.) $\left.\begin{array}{l} 3x-2y=46 \\ 9x+5y=215 \end{array}\right\}.$ (12.) $\left.\begin{array}{l} 3(x+y)=90 \\ 5x-2y=59 \end{array}\right\}.$

(13.) $\left.\begin{array}{l} x+y-z=4 \\ x-y+z=2 \\ y+z-x=0 \end{array}\right\}.$ (14.) $\left.\begin{array}{l} 2x+3y+z=52 \\ 4x-2y+3z=40 \\ 5x-2y-5z=20 \end{array}\right\}.$

(15.) $\left.\begin{array}{l} 22x+31y-16z=279 \\ 13x-17y+12z=-18 \\ 15x-3y-5z=39 \end{array}\right\}.$

(16.) Divide 30 into two parts such that one shall be four times the other.

(17.) Find a number such that six times its half are equal to five times its third part.

(18.) Divide £10. 10s. between A and B, so that A may receive $\frac{3}{4}$ of what B receives.

(19.) A and B working together can do a piece of work in 13 days. A works three times as hard as B; in what time could he do the work alone?

(20.) I buy a spade, a rake, and a hoe, for 12s. 6d., and give 3d. more for the rake than the hoe, but 4s. more for the spade than the rake. What did each cost?

(21.) Into a cistern two thirds full are poured 12 gallons; and it is then three fourths full. How many gallons will it hold?

(22.) A certain fraction becomes $\frac{1}{2}$ when 1 is added to its denominator, and 1 when 1 is added to its numerator. What is the fraction?

(23.) A and B play for 5s. If A wins, he has as much as B; but if he loses, he has only a third as much as B. How much has each of them?

(24.) There is a number of two digits. If the first, or tens' digit, be increased by 5, the number is made three times larger. If the second, or units' digit, be increased by 3, the number is equal to four times the sum of the original digits. What is the number?

(25.) A and B working together could finish a piece of work in k days. They worked together c days, when A was called away, and B finished the work in p days. In what time could each do it alone?

(26.) A person after paying a poor-rate, and also an income-tax of 7d. in the pound, has £486 remaining. The poor-rate is £22. 10s. more than the income-tax. Find the original income, and the number of pence per pound in the poor-rate.

(27.) When are the hands of a clock together between 3 and 4?

(28.) There are two numbers whose sum is $3a$, and their difference a. Find them.

(29.) Find three numbers such that the first is to the second, as the second to the third; the third is 9 times the first, and the sum of the first and second is 12.

(30.) There is a number of four digits: the sum of the digits is 21; the third digit is twice the first; the sum of the first and last is twice the second; and if 4086 be added to the number, the digits will be inverted. What is the number?

APPENDIX.

A. To prove the rule for finding the G.C.M. of two expressions. (See *Arith.* § 35 ; *Alg.* § 32.)

Let X, Y, denote the two expressions. Divide Y by X, and let p denote the quotient, and Z the remainder, that is, $Y-pX$. Divide X by Z, and let q denote the quotient, and $W = X - qZ$, the remainder. Repeat this process until there is no remainder. Suppose $W(= X - qZ)$, the last divisor.

$$\begin{array}{l} X)\,Y\,(p \\ \quad \underline{pX} \\ \quad Z)\,X\,(q \\ \qquad \underline{qZ} \\ \qquad W)\,Z\,(r \\ \qquad\quad \underline{rW} \end{array}$$

Then W divides Z without remainder; therefore it must divide $W+qZ$, or X, without remainder; and therefore also it must divide $Z + pX$, or Y, without remainder. Hence W is a common measure of X and Y.

And as W is the greatest measure of itself, and as every measure of X and Y must be a measure of $Y-pX$, or Z, and therefore of $X - qZ$, or W, it follows that W is the *greatest* common measure of X and Y.

B. To illustrate the rule for converting a vulgar fraction into a decimal. (*Arith.* § 16.)

Let $\frac{a}{b}$ represent the vulgar fraction; x, the significant figures of the decimal; y, the number of 0's assumed in the division.

$$\text{Then } \frac{a}{b} = \frac{x}{(10)^y}$$

$$\therefore x = \frac{a \times (10)^y}{b}$$

Hence, if the numerator be multiplied by any power of 10, and this product be divided by the denominator, the quotient will give the significant figures of the decimal, and the number of decimal places will be the same as the index of the power of 10.

C. To prove the rule for finding a vulgar fraction equivalent to a given recurring decimal. (*Arith.* § 57.)

Let a represent the non-recurring decimal figures, and let m = their number.

Let k represent the recurring figures, and let n = their number.

Let F represent the whole decimal, and let x = its value.

Then $\quad x = F,$

$$(10)^n x = k + \text{the recurring period},$$
$$(10)^m x = a + \text{the recurring period}.$$

Hence, $\quad (10)^n - (10)^m \quad x = k - a;$

$$\text{or,} \quad x = \frac{k - a}{(10)^n - (10)^m}.$$

Now $(10)^n - (10)^m$, where n is greater than m, will always consist of $n - m$ 9's, followed by m 0's.

Hence the rule: Take for the numerator of the vulgar fraction, the figures to the end of the first period, diminished by the non-circulating part; and for the denominator, as many 9's as there are circulating figures, followed by as many 0's as there are non-circulating figures.

EXAMPLE.—·32689

$$\begin{aligned} \text{Let} \quad x &= \cdot 32689 \quad 689\ldots\ldots \\ \text{then } 100000\,x &= 32689 \cdot 689\ldots\ldots \\ 100\,x &= 32 \cdot 689\ldots\ldots \\ \text{hence, } (100000 - 100)\,x &= 32689 - 32 \\ x &= \frac{32657}{99900}. \end{aligned}$$

D. Annuities.—(*Arith.*, § 96.)—By aid of algebraical symbols we can find a formula which enables us to find the present value of any annuity, whether the interest be reckoned as simple or compound. In both cases, however, we have to assume results beyond the stage to which we are limited in this volume. We will merely give the formula when the interest is reckoned as simple.

Here we must assume that the sum of the first n natural numbers $(1, 2, 3 \dots n) = \frac{n(n+1)}{2}$.

Now let P denote the value of the annuity, which consists of A pounds, payable yearly for n years; and let r denote the *ratio of increase* of the simple interest.

(Thus, if interest be reckoned at 5 per cent., $r = \frac{5}{100} = \frac{1}{20}$.)

Then the simple interest on P pounds for n years, at the rate r, $= Pnr$.

And as the interest must be reckoned on each payment of the annuity except the last, the sum of these interests will be $(1 + 2 + 3 + \dots + \overline{n-1})\, rA = \frac{n(n-1)}{2} rA$, as assumed.

$$\text{Hence, } P + Pnr = nA + \frac{n(n-1)}{2} rA;$$

$$\text{or, } P = \frac{nA + \frac{1}{2}n(n-1)rA}{1 + nr}.$$

Suppose the annuity to be £100, for 100 years, at 5 per cent.

$$\text{Then, } P = £100\left(\frac{100 + 50 \times 99 \times \frac{1}{20}}{1 + 5}\right)$$

$$= £100\,\frac{200 + 495}{12} = £25 \times \frac{695}{3},$$

$$= \frac{£17375}{3},$$

$$= £5791.\ 13s.\ 4d.$$

SOLUTIONS OF THE QUESTIONS

SET AT THE GOVERNMENT EXAMINATIONS OF SCIENCE SCHOOLS,

FROM 1867 TO 1872.

1867.

[In this and the following year, students were divided into five classes, and a paper of twelve questions was set to the lowest three. Of these questions the following three only were of a nature corresponding to the present First Stage.]

2. Solve the equation $\frac{x + 2}{3} + \frac{x + 3}{2} = x$;
and the equation $(3x - 1)^2 + (4x - 2)^2 = (5x - 3)^2$.

(i.) L.C.M. = 6. Multiply by 6 throughout.

$$\text{Then } 2(x + 2) + 3(x + 3) = 6x;$$
$$\text{or, } 2x + 3x - 6x = -4 - 9, \text{ \&c.}$$

(See *Alg.* § 45.)

(ii.) Square the binomial expressions.

$$\text{Thus, } (9x^2 - 6x + 1) + (16x^2 - 16x + 4)$$
$$= (25x^2 - 30x + 9).$$
$$\text{or, } 25x^2 - 22x + 5 = 25x^2 - 30x + 9$$
$$-22x + 30x = 9 - 5, \text{ \&c.}$$

(See *Alg.* § 46.)

3. Find the least common multiple of 16, 24, and 30, and explain the method.

The greatest common measure of 16 and 24 = 8,

$\therefore$ the least common multiple of 16 and 24 $= \frac{16}{8} \times 24$
$= 2 \times 24 = 48.$

The greatest common measure of 48 and 30 = 6,

$\therefore$ the least common multiple of 48 and 30, that is, of 16, 24, and 30, $= \frac{48}{6} \times 30 = 8 \times 30 = 240.$

Or,

$$\left.\begin{array}{l} 16 = 2 \times 2 \times 2 \times 2 \\ 24 = 2 \times 2 \times 2 \times 3 \\ 30 = 2 \times 3 \times 5 \end{array}\right\} \begin{array}{l} \therefore \text{ L.C.M. of 16, 24, and 30} \\ = (2 \times 2 \times 2 \times 2) \times 3 \times 5 \\ = 16 \times 3 \times 5 = 240. \end{array}$$

(For the explanation of the methods, see *Arith.* §§ 36, 37.)

5. Find the square root of 26 to four places of decimals.

$$\begin{array}{r|l} & \dot{2}\dot{6}(5{\cdot}099 \\ & 25 \\ \hline 1009 & 10000 \\ & 9081 \\ \hline 10189 & 91900 \\ & 91701 \\ \hline & 199 \end{array}$$

And similarly, to find one more figure of the root.

(See *Arith.* §§ 60, 61.)

1868.

[In this year twelve questions were set to the three lowest grades; and out of these the following five corresponded to the present First Stage.]

1. Calculate to four places of decimals the value of this expression :—

$$\frac{\frac{5}{8} \text{ of } {\cdot}31416}{\sqrt[3]{{\cdot}93}}.$$

(i.) *By vulgar fractions partially.*

$\frac{5}{8}$ of $\cdot 31416 = \frac{5}{8} \times \frac{31416}{100000} = \frac{3927}{20000}$

$\sqrt{\cdot 93} = \sqrt{\frac{93}{100}} = \frac{9\cdot 643650}{10}$

$\therefore \dfrac{\frac{5}{8} \text{ of } \cdot 31416}{\sqrt[2]{\cdot 93}} = \frac{3927}{20000} \times \frac{10}{9\cdot 643650}$

$= \frac{3297}{192843}$,

and the numerator must be divided by the denominator.

$$
\begin{array}{r|l}
 & 9\dot{3}(9\cdot 643650 \\
 & 81 \\
\hline
186 & 1200 \\
 & 1116 \\
\hline
1924 & \quad 8400 \\
 & \quad 7696 \\
\hline
19283 & \quad\; 70400 \\
 & \quad\; 57849 \\
\hline
192866 & \quad 1255100 \\
 & \quad 1157196 \\
\hline
1928725 & \quad\; 9790400 \\
 & \quad\; 9643625 \\
\hline
 & \quad\quad 146775
\end{array}
$$

(ii.) *By decimals throughout.*

$\frac{5}{8}$ of $\cdot 31416 = 5 \times \cdot 03927 = \cdot 19635$.

$\sqrt[2]{\cdot 93} = 9\cdot 643650$, as shown above.

$\therefore \dfrac{\frac{5}{8} \text{ of } \cdot 31416}{\sqrt[2]{\cdot 93}} = \frac{\cdot 19635}{9\cdot 64365}$.

And the numerator must be divided by the denominator.

(The latter method is generally preferable, except where circulating decimals enter into the expression. It is always necessary to take two or three more decimal places in a fraction than the number required in the answer. In the extraction of $\sqrt{93}$ above, we have taken six; and as the sixth decimal figure is 0, we may be sure that by taking the previous *five* we shall obtain a correct result. (See *Arith.* §§ 60, 61).

2. Solve the equations:—

(1.) $5(4 - x) + 3(x - 6) = 10$.

(2.) $(x + 3)^2 - 3x(4x + 1) = 5x^2 - (4x - 5)^2$.

(1.) Since $5(4 - x) + 3(x - 6) = 10$,
$$20 - 5x + 3x - 18 = 10.$$
$$\text{or, } -5x + 3x = 10 - 20 + 18, \text{ \&c.}$$

(2.) Since $(x + 3)^2 - 3x(4x + 1) = 5x^2 - (4x - 5)^2$,
$$x^2 + 6x + 9 - 12x^2 - 3x = 5x^2 - 16x^2 + 40x - 25,$$
$$\text{or, } -11x^2 + 3x + 9 = -11x^2 + 40x - 25,$$
$$\text{or, } 3x - 40x = -25 - 9, \text{ \&c.}$$

(See *Alg.* § 46).

3. Find the greatest common measure of 256, 116.

```
116)256(2
    232
     24)116(4
         96
         20, &c.
```

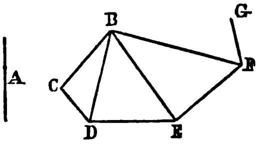
(See *Arith.* § 35.)

5. Extract the square root of 407 to two places of decimals.

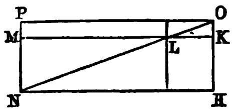
```
     4̇0̇7(20·1
     4
401 | 700
    | 401
      299, &c.
```

(See *Arith.* § 61.)

6. Show how to find a rectangular parallelogram, which shall have one side equal to a given line, and which shall be equal in area to a given rectilineal figure.



Let A be the given line, and BCDEFG... the given rectilineal figure.



Constr.—Join BD, BE, BF... Make a rectangular parallelogram HKL equal to the triangle BCD. Produce KL to M, making LM = A; and complete the parallelogram HM. Join NL, and produce it to meet HK produced in O.

Complete the parallelogram HOPN.

Then, since the complements are equal, LP = LH = ABCD.

By the like method, apply to LM a parallelogram = BDE, and to successive parallel sides apply parallelograms equal to BEF, BFG, &c., until the rectilineal figure BCDEFG... is exhausted.

Then the whole rectangular parallelogram PL... shall be the one required.

(Proof as in *Geometry*, Prop. 45.)

1869.

[From this year the papers for the First Stage have been set separately.]

1. Explain the method of turning a vulgar fraction into a decimal. (See *Arith.*, § 16.)

Find the decimal fraction equivalent to $\frac{16}{113}$ to four places. (10.)

$$\begin{array}{r}
113)16\ (\cdot14\ldots \\
\underline{113} \\
470 \\
\underline{452} \\
18,\ \&c.
\end{array}$$

* 2. Extract the square root of $141\cdot7$ to two places of decimals. (10.)

$$\begin{array}{r|l}
 & \dot{1}4\dot{1}\cdot7(11\cdot8 \\
 & 1 \\
21 & \overline{41} \\
 & \underline{21} \\
228 & 2000 \\
 & \underline{1824} \\
 & 176,\ \&c.
\end{array}$$

(See *Arith.*, § 61.)

* 3. Show that parallelograms and triangles upon the same base and between the same parallels are equal to one another. (10.)

(This appears to mean that parallelograms on the same base and between the same parallels are equal, and also that triangles on the same base and between the same parallels are equal. See *Geometry*, Props. 35, 37.)

* 6. Divide $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$ by $a^3 - 3a^2x + 3ax^2 - x^3$. (10.)

$$
\begin{array}{l}
a^3 - 3a^2x + 3ax^2 - x^3)\,a^6 - 3a^4x^2 + 3a^2x^4 - x^6\,(a^3 + 3a^2x + 3ax^2 \\
\qquad\qquad\qquad\qquad\quad \underline{a^6 - 3a^5x + 3a^4x^2 - a^3x^3} \\
\qquad\qquad\qquad\qquad\qquad 3a^5x - 6a^4x^2 + a^3x^3 + 3a^2x^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{3a^5x - 9a^4x^2 + 9a^3x^3 - 3a^2x^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 3a^4x^2 - 8a^3x^3 + 6a^2x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3a^4x^2 - 9a^3x^3 + 9a^2x^4 - 3ax^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad —\&c.
\end{array}
$$

7. Solve the equations :—

* (1.) $\dfrac{5x - 6}{9} - \dfrac{2x - 13}{3} = \dfrac{x + 7}{3}$. (8.)

(2.) $x - \dfrac{x - 2}{3} = 5\frac{3}{4} - \dfrac{10 + x}{5} + \dfrac{x}{4}$. (8.)

(1.) Since $\dfrac{5x - 6}{9} - \dfrac{2x - 13}{3} = \dfrac{x + 7}{3}$, multiply throughout by the L.C.M., 9.

Then $5x - 6 - 3(2x - 13) = 3(x + 7)$

$5x - 6x - 3x = 21 + 6 - 39$, &c. (*Alg.*, § 45.)

(2.) Since $x - \dfrac{x - 2}{3} = \dfrac{23}{4} - \dfrac{10 + x}{5} + \dfrac{x}{4}$, multiply throughout by the L.C.M., 60.

Then $60x - 20(x - 2) = 345 - 120 - 12x + 15x$;
or, $60x - 20x + 12x - 15x = 345 - 120 - 40$, &c.

(*Alg.*, § 45.)

(The remaining questions of this paper—seven—are beyond the present requirements of the First Stage. Those marked with an asterisk were recommended to be

attempted first, as being the simplest. The numbers after the questions represent the value given for a correct solution.)

1870.

[INSTRUCTIONS.

You are only permitted to attempt *eight* questions. You may select these from any part of the paper. The value attached to each question is the same (12), except the two last, which are worth (16), only one of which may be taken.]

1. Find the numerical value of $\dfrac{a^2+\sqrt{b^2-a}}{a^2-\sqrt{b^2-a^2}}+\dfrac{c\sqrt{c^2+1}}{b\sqrt{a^2+9}}$, when $a=5$, $b=6$, $c=4$, to two places of decimals.

When $a=5$, $b=6$, $c=4$,

$$\frac{a^2+\sqrt{b^2-a^2}}{a^2-\sqrt{b^2-a^2}}+\frac{c\sqrt{c^2+1}}{b\sqrt{a^2+9}}=\frac{25+\sqrt{36-25}}{25-\sqrt{36-25}}+\frac{4\sqrt{16+1}}{6\sqrt{34}},$$

$$=\frac{25+\sqrt{11}}{25-\sqrt{11}}+\frac{2\sqrt{17}}{3\sqrt{34}}$$

$$=\frac{25+\sqrt{11}}{25-\sqrt{11}}+\frac{\sqrt{2}}{3}\left(\text{since } \frac{2\sqrt{17}}{3\sqrt{34}}=\frac{\sqrt{2}.\sqrt{2}.\sqrt{17}}{3.\sqrt{2}.\sqrt{17}}\right).$$

$$\left[\begin{array}{l}11(3\cdot31662\\9\end{array}\right]=\frac{28\cdot31662}{21\cdot68338}+\frac{\sqrt{2}}{3},\ \&c.$$

$$\begin{array}{r|l}
63 & \overline{200} \\
 & 189 \\
661 & \overline{1100} \\
 & 661 \\
6626 & \overline{43900} \\
 & 39756 \\
66326 & \overline{414400} \\
 & 397956 \\
663262 & \overline{1644400}
\end{array}$$

&c. (See *Alg.*, §§ 1—13; *Arith.*, § 61.)

2. Convert $\frac{1}{128}$ into a decimal fraction; and find the vulgar fraction corresponding to the recurring decimal $\cdot 22\dot{2}9\dot{7}$.

(1.) $128)1{\cdot}000(\cdot 007$

$$\begin{array}{r} 896 \\ \hline 104, \text{ \&c.} \end{array}$$

(2.) $$\cdot 22\dot{2}9\dot{7} = \frac{22297 - 22}{99900}, \text{ \&c.}$$

(See *Arith.*, §§ 16, 57.)

3. Find the G.C.M. of $48x^2 + 16x - 15$ and $24x^3 - 22x^2 + 17x - 5$; and the L.C.M. of $4k^2 - 9y^2$, $4k^2 - y^2$, $4k^2 - 8ky + 3y^2$, $4k^2 + 4ky - 3y^2$, $4k^2 - 4ky - 3y^2$, $4k^2 + 8ky + 3y^2$.

(1.) $24x^3 - 22x^2 + 17x - 5$

$$\begin{array}{r|l} & 2 \\ \therefore 48x^2 + 16x - 15 & 48x^3 - 44x^2 + 34x - 10\,(x - 5 \\ & 48x^3 + 16x^2 - 15x \\ \hline & \quad - 60x^2 + 49x - 10 \\ & \qquad\qquad\qquad 4 \\ \hline & -240x^2 + 196x - 40 \\ & -240x^2 - 80x + 75 \\ \hline 23 & 276x - 115 \\ \hline & 12x - 5 \end{array}$$

$12x - 5)48x^2 + 16x - 15($ &c. (See *Alg.*, § 32).

(Here the multiplication of the first dividend by 2, and of the partial dividend by 4, and the subsequent division by 23, were necessary in order to avoid the occurrence of fractions.)

(2.)
$$\begin{aligned} 4k^2 - 9y^2 &= (2k + 3y)(2k - 3y). \\ 4k^2 - y^2 &= (2k + y)(2k - y). \\ 4k^2 - 8ky + 3y^2 &= (2k - 3y)(2k - y). \\ 4k^2 + 4ky - 3y^2 &= (2k + 3y)(2k - y). \\ 4k^2 - 4ky - 3y^2 &= (2k - 3y)(2k + y). \\ 4k^2 + 8ky + 3y^2 &= (2k + 3y)(2k + y). \\ \therefore \text{L.C.M.} &= (2k + 3y)(2k - 3y)(2k + y)(2k - y) \end{aligned}$$

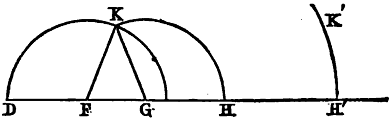
&c. (See *Alg.*, § 33.)

4. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles. Prove this. (See *Geom.*, Prop. 14.)

5. Show how to construct a triangle of which the sides shall be equal to three given straight lines.
(See *Geom.*, Prop. 20.)

Why must any two be greater than the third? Show by figure what will happen if this condition be not observed.

It was proved in a previous proposition (*Geom.*, Prop. 18) that any two sides of a triangle must be greater than the third.



If this condition be not observed, if, for instance (taking the figure of Prop. 20), GH′ be greater than GF and FD together, that is, greater than GD, it is manifest that the circles DK, H′K′ will never meet, as the construction assumes them to do.

If GH = GD, the circles will meet in D, and the triangle becomes a straight line.

A like thing would happen if DF or FG were greater than the other two together.

6. All the interior angles of a triangle are equal to two right angles. Prove this. (*Geom.*, Prop. 31.)

What is the sum of all the interior angles in a figure of seven sides?

Prop. 32 shows that all the inner angles of a figure, together with 4 right angles = twice as many right angles as the figure has sides.

Here the figure has 7 sides.

Therefore, the inner angles + 4 right angles = 14 right angles, or, the inner angles alone = 10 right angles.

7. Raise $\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3$ to its third power, by multiplication.

$$\begin{array}{r}
\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 \\
\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 \\
\hline
\frac{1}{4}x + \frac{1}{6}x^3 - \frac{1}{8}x^4 \qquad\qquad\qquad\qquad \\
+ \frac{1}{6}x^3 + \frac{1}{9}x^4 - \frac{1}{12}x^5 \qquad\qquad \\
- \frac{1}{8}x^4 - \frac{1}{12}x^5 - \frac{1}{16}x^6 \\
\hline
\frac{1}{4}x + \frac{1}{3}x^3 - \frac{5}{36}x^4 - \frac{1}{6}x^5 + \frac{1}{16}x^6
\end{array}$$

And, similarly, one more step. (*Alg.*, §§ 23, 30).

Or, $\frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 = \frac{1}{12}(6x + 4x^2 - 3x^3)$; and we may cube the expression in the bracket, and prefix $\frac{1}{(12)^3}$ or, $\frac{1}{1728}$.

But in examination, it would be better to employ the first method, as that is clearly what is desired.

8. Prove that $a^m \times a^n = a^{m+n}$ when (m) (n) are positive whole numbers; show that the same rule holds when either (m) or (n), or both, are negative integers; and show that $a^0 = 1$. (See *Alg.*, §§ 5, 21, 24, 26.)

9. Solve *two* of the following equations:—

(1.) $\frac{1}{14}(3x + \frac{2}{3}) - \frac{1}{7}(4x - 6\frac{2}{3}) = \frac{1}{2}(5x - 6)$.

(2.) $\sqrt{a + x} + \sqrt{a - x} = 2\sqrt{x}$.

(3.) $\frac{1}{3} - \frac{7x - 1}{6\frac{1}{2} - 3x} = \frac{8}{3} \cdot \frac{x - \frac{1}{2}}{x - 2}$.

(1.) Multiply by 14 and 3, then $9x + 2 - 24x - 40 = 105x - 126$, &c.

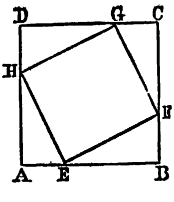
(2.) Squaring, $a + x + a - x + 2\sqrt{a^2 - x^2} = 4x$; $a - 2x = -\sqrt{a^2 - x^2}$; squaring again, $a^2 - 4ax + 4x^2 = a^2 - x^2$; $5x^2 = 4ax$, &c.

(3.) $-\frac{14x - 2}{13 - 6x} = \frac{8(x - \frac{1}{2}) - (x - 2)}{3(x - 2)} = \frac{7x - 2}{3x - 6}$.

Or, $(14x - 2)(3x - 6) = (7x - 2)(6x - 13)$, &c. (*Alg.*, § 46).

10. Describe a square upon a given straight line, and then inscribe another square within it, having one of its angles at a given point in the side.

The first part of the problem consists of *Geom.*, Prop. 46.



Let AB be the given line; ABCD, the square; E, the given point in AB.

Suppose the square EFGH to be inscribed in ABCD as required.

Then in the triangles AEH, BFE, —since the angles AEH, FEB together = a right angle (Prop. 9), and also the angles AEH, EHA together = a right angle (Prop. 31)—the angle EAH = the angle BFE, and the angle A = the angle B; also EH, FE, opposite equal angles, are equal; therefore the triangle AEH = the triangle BFE (Prop. 24).

Hence, AE = BF.

Now, therefore, we can give the following construction for the second part of the problem :—

Cut off from BC, CD, DA, parts equal to AE; then E, F, G, H, the points thus found, will be the angular points of the inscribed square.

(This process of assuming that a problem has been solved, and deducing from the figure a method of construction, is called *analysis*, and is often useful in the solution of problems.)

11. Reduce $\dfrac{\sqrt[3]{5 \cdot 12} + \sqrt[3]{\cdot 03375}}{\sqrt[3]{80} - \sqrt[3]{1}}$ to an equivalent single decimal.

Multiply each term of the complex fraction by $\sqrt[3]{100}$; it then becomes

$$\frac{\sqrt[3]{512} + \sqrt[3]{3 \cdot 375}}{\sqrt[3]{8000} - \sqrt[3]{1}}$$

$$= \frac{8 + 1 \cdot 5}{20 - 1}, \text{ by extraction of the roots,}$$

$$= \frac{9 \cdot 5}{19}, \text{ \&c.}$$

(The multiplication by $\sqrt[3]{100}$ was suggested by the appearance of the surd quantities $\sqrt[3]{80}$ and $\sqrt[3]{\cdot 01}$; for 8000 and 1 are manifestly cubes.

See *Alg.*, § 28.)

Or,

12. Show that if $x = \left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$

then $\frac{1}{2}\frac{a^2-b^2}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x}) = \left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}$

$$\frac{1}{2}\frac{a^2-b}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x}) = \frac{1}{2}\frac{a^2-b^2}{a^2+b^2}\left\{\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}+\left(\frac{a+b}{a-b}\right)^{\frac{2p}{q-p}}\right\}$$

$$= \frac{1}{2}\frac{a^2-b^2}{a^2+b^2}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}\left\{1+\left(\frac{a+b}{a-b}\right)^{\frac{2(p-q)}{q-p}}\right\}$$

$$= \frac{1}{2}\frac{a^2-b^2}{a^2+b^2}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}\left\{1+\left(\frac{a+b}{a-b}\right)^{-2}\right\}$$

$$= \frac{1}{2}\frac{a^2-b^2}{a^2+b^2}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}}\cdot\frac{2(a^2+b^2)}{(a+b)^2}$$

$$= \frac{a-b}{a+b}\cdot\left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}} = \left(\frac{a+b}{a-b}\right)^{\frac{2q}{q-p}-1} = \left(\frac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}.$$

1871.

[INSTRUCTIONS.

No student will obtain a class unless he gains fair marks in two of the three subjects, Arithmetic, Algebra, Geometry.

Not more than eight questions are to be answered, of which there must be at least two in Arithmetic, two in Algebra, and two in Geometry.

The value attached to each question is the same.]

1. Find the numerical value of $\frac{a(b-\sqrt{c})^2}{\sqrt{b^2-a^2}}$, when $a=3$, $b=4$, $c=5$, to two places of decimals.

$$\frac{a(b = \sqrt{c})^2}{\sqrt{b^2 - a^2}} = \frac{3\,(4 - \sqrt{5})^2}{\sqrt{16 - 9}} = \frac{3\,(16 - 8\sqrt{5} + 5)}{\sqrt{7}}.$$

$$\left[\begin{array}{ll} 7(2{\cdot}6457\ldots & 35(5{\cdot}9160\ldots \\ 4 & 25 \end{array}\right] = \frac{63 - 8\sqrt{5}}{\sqrt{7}}$$

46	300	109	1000
	276		981
524	2400	1181	1900
	2096		1181
5285	30400	11826	71900
	26425		70956
	3975		1944

$$= \frac{63\sqrt{7} - 8\sqrt{35}}{7}$$

$$= 23{\cdot}8113 - 6{\cdot}7608, \text{ \&c.}$$

(Here it will be noticed that if we had not taken four figures of the root of 7, we should have had 23·80... in place of 23·81, which would have given an incorrect result. Whenever the digit next after the number required in the answer is 0, as in 6·7608, we need not take account of the next digit.)

(See *Arith.*, § 61.)

2. Find two numbers, whose sum shall be 8, and their difference ·6.

$$\begin{aligned} \text{Let } x &= \text{one number,} \\ \text{then } 8 - x &= \text{the other.} \\ \therefore\ x - (8 - x) &= {\cdot}6, \\ 2x &= 8{\cdot}6, \text{ \&c.} \end{aligned}$$

(*Alg.*, § 47, 48.)

3. Divide $x^8 - xy^7 + y^8$ by $x^2 - xy + y^2$.

$$x^2 - xy + y^2)\,x^8 - xy^7 + y^8\,(x^6 + x^5y - x^3y^3 - x^2y^4, \text{ \&c.}$$

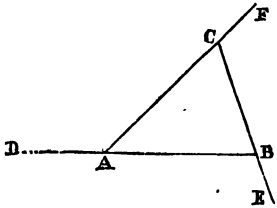
$$\begin{array}{l} \underline{x^8 - x^7y + x^6y^2} \\ \quad x^7y - x^6y^2 \\ \quad \underline{x^7y - x^6y^2 + x^5y^3} \\ \qquad - x^5y^3 \\ \qquad \underline{- x^5y^3 + x^4y^4 - x^3y^5} \\ \qquad\quad - x^4y^4 + x^3y^5 \\ \qquad\quad \underline{- x^4y^4 + x^3y^5 - x^2y^6} \\ \qquad\qquad\qquad \text{\&c.} \end{array}$$

4. Define a right angle, parallel lines, square, parallelogram, diagonal, complement. (See *Geometry*, Definitions 9, 29, 25, 30, 32, and 37.)

5. If two triangles have two sides of the one equal to two sides of the other, and likewise their bases equal, the triangles shall be equal in every respect. (See *Geom.*, Prop. 4.)

6. The interior angles of a triangle are together equal to two right angles. Prove this. (*Geom.*, Prop. 31.)

Define the exterior angle of a triangle, and, according to your definition, find the sum of the exterior angles.



Def.—If any side of a triangle be produced, the angle formed adjacent to the interior angle is called exterior. By *the sum of the exterior angles* is meant the sum of the three exterior angles made by producing the sides successively in the same direction. Let the sides BA, AC, CB, of the triangle ABC, be produced to D, F, E.

Then every exterior angle, together with its adjacent interior angle, is equal to two right angles (Prop. 9).

Therefore the three exterior angles, together with the three interior angles, are equal to six right angles.

But, by the Proposition just proved, the three interior angles = two right angles.

Therefore the sum of the exterior angles of a triangle = four right angles.

(The student should always remember to pay close attention to the wording of a question. In this case we are asked to solve the problem "according to our definition." Now, without the second part of the definition above given, the sides might have been supposed to be produced in both directions, and we should have had *six* adjacent exterior angles; the sum of which would be eight right angles.)

7. Parallelograms on the same base and between parallels are equal to one another. (*Geom.*, Prop. 35.)

8. Explain clearly why the product of a negative quantity, in algebra, by a negative quantity must have the positive sign. (*Alg.*, § 22.)

9. Solve *two* of the following equations :—

(1.) $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$.

(2.) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$.

(3.) $\frac{2x-x^2}{(1-x)^2}=a$.

(4.) $\sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{x+1}}$.

(1.) $2(x-5)+3x=72-2(x-10)$
$2x+3x+2x=72+20+10$, &c.

(2.) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{9}{28}$

$$=\frac{16x+24}{28}$$

$$=\frac{4x+6}{7},$$

or, $\frac{7x-3}{6x+2}=\frac{7}{14}=\frac{1}{2}$,

$\therefore\quad 7x-3=3x+1$, &c.

(3.) A *quadratic.*

(4.) $\sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{x+1}}$.

$\therefore\quad x+1+\sqrt{x^2-1}=2$

$\sqrt{x^2-1}=1-x$

or, squaring, $x^2-1=1-2x+x^2$, &c.

10. Reduce $\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)$ $+\frac{a+c}{ac}(a^2+c^2-b^2)$, to its simplest form.

The expression $=\left(\frac{1}{b}+\frac{1}{a}\right)(a^2+b^2-c^2)+\left(\frac{1}{c}+\frac{1}{b}\right)$ $(b^2+c^2-a^2)+\left(\frac{1}{c}+\frac{1}{a}\right)(a^2+c^2-b^2)=a+b+\frac{a^2}{b}+\frac{b^2}{a}$ $-\frac{c^2}{b}-\frac{c^2}{a}+c+b+\frac{b}{c}+\frac{c^2}{b}-\frac{a^2}{c}-\frac{a^2}{b}+c+a+\frac{a^2}{c}+\frac{c^2}{a}$ $-\frac{b^2}{c}-\frac{b^2}{a}$; and the fractions disappear.

11. A foreman has 116 half-crowns, and wants to pay 50 men, some at 6s. 6d., and some at 5s. 6d. He finds he can do this without change by paying them in groups. How many are there in each group?

Suppose there are x men in one group,
then there are $50-x$ in the other.
The x men receive $13\,x$ sixpences (in half-crowns),
the $50-x$ men receive $11(50-x)$ sixpences (in half-crowns).
$\therefore\ 13\,x+11\,(50-x)=580$, &c. (*Alg.*, § 47.)

12. Reduce $\frac{\sqrt{1{\cdot}75}-\sqrt{{\cdot}63}}{\sqrt{3{\cdot}5}-\sqrt{2{\cdot}1}}$ to its simplest form.

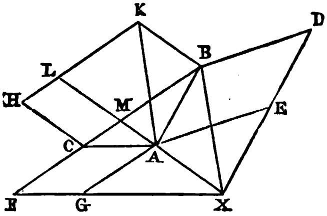
Divide each term by $\sqrt{7}$; then the expression becomes

$$\frac{\sqrt{{\cdot}25}-\sqrt{{\cdot}09}}{\sqrt{{\cdot}5}-\sqrt{{\cdot}3}}$$
$$=\sqrt{{\cdot}5}+\sqrt{{\cdot}3} \qquad (\textit{Alg.}, \S\ 23, \text{iii.})$$

which can be easily worked out.

13. In any triangle parallelograms are described upon AB, AC, the sides containing the angle A. If the sides parallel to AB, AC be produced to meet in X, and AX

be joined, a parallelogram described on BC, with its sides parallel and equal to AX, will be equal to the two parallelograms described upon the sides.



Let ABDE, ACFG, BCHK be the parallelograms. Produce XA to meet HK in L; and join AK, BX.

Then BKLM = BKAX, because BK, LX are parallel, and BKAX is a parallelogram. (*Geom.*, Prop. 33 *and* 35.)

And BKAX is double of the triangle BAX.

And therefore = BAED, because BA, DX are parallel.

Hence BKLM = BAED.

Similarly MCHL = ACFG.

Therefore the whole BKHC = the two BAED and ACFG.

1872.

[INSTRUCTIONS.

The paper is divided into three sections. You are not to answer more than eight questions. No student can be passed who omits altogether one section.]

SECTION I.—ARITHMETIC.

1. Show by an easy example that division of one whole number by another is equivalent to a series of subtractions.

Divide 1·02 by $\frac{1\frac{7}{8}}{3\frac{3}{4}}$ of ·144.

(i.) $23 \div 4 = 5 + (3 \div 4)$.

By *Subtraction*—

$23 - 4(=19) - 4(=15) - 4(=11) - 4(=7) - 4(=3)$;
which shows that 4 is contained in 23 five times.

By *Division*—

$$\begin{array}{r} 4)\overline{23}(1 \\ 4 \\ \hline 4)\overline{19}(1 \\ 4 \\ \hline 4)\overline{15}(1 \\ 4 \end{array} \qquad \begin{array}{r} 4)\overline{11}(1 \\ 4 \\ \hline 4)\overline{\ 7}(1 \\ 4 \\ \hline 3 \end{array}$$

2. If a parcel of 12 lbs. weight is carried 80 miles by rail for 2s. 4d., and the rate for the distance over 50 miles is two-thirds of the rate for the first 50 miles, how far can a parcel of 8 lbs. be carried for 4d.?

2s. 4d. = charge for 50 miles;
+ $\frac{2}{3}$ of $\frac{3}{5}$ of the charge for 50 miles;
or, 28d. = $\frac{7}{5}$ of the charge for 50 miles;
∴ 4d. = $\frac{1}{5}$ " "
and 20d. = the charge for the first 50 miles;
that is, 12 lbs. are carried 50 miles for 20d.;
∴ 1 lb. is carried 600 miles for 20d. (*at the same rate*);
∴ 8 lbs. are carried 75 miles for 20d.;
∴ 8 lbs. are carried 15 miles for 4d.

3. If the 3 per cents. are at $91\frac{1}{8}$, what interest does this give on £100? (Omit brokerage and fractions of a penny).

$$\text{Interest} = £\tfrac{3}{100} \times \frac{100}{91\frac{1}{8}} \times 100.$$

4. How many lbs. in ·321875 of a ton weight? Convert it into kilogrammes (omitting fractions), assuming that a cubic decimetre of distilled water weighs 15432·35 grains.

(i.) Multiply ·321875 by $20 \times 4 \times 28$.
(ii.) See *Metrical System*, p. 54.

Section II.—Geometry.

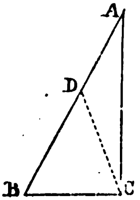
5. Define an *angle;* and show how to divide a given angle, traced upon paper, into two equal parts, with a ruler and compass; and show that your method is a true one. (See *Geometry*, Def. 7 and Prop. 5.)

6. Show that any two sides of a triangle must together be greater than the third side. Given three straight

lines, make a triangle whose sides shall be equal to them; and show how the construction fails in certain cases.

(See *Geometry*, Prop. 18; also question 5 for 1870.)

7. Define a *right angle;* show how, mechanically, to prove that the two acute angles of a right-angled triangle are together equal to a right angle, by an independent method. (See *Geometry*, Def. 9.)



(ii.) Cut a piece of paper, ABC, in the form of a right-angled triangle, C being the right angle. Bisect AB in D. Fold the paper so that A falls on C. Then DC = DA = DB. Therefore, DBC is an isosceles triangle, and angle DBC = angle DCB. And the acute angles at A and B = DCA + DBC = the right angle C.

8. Define a *rhombus* and its *diagonals*. Prove that the diagonals of a rhombus bisect each other at right angles.

(See *Geom.*, Def. 26 and 32.)

(ii.) The diagonals bisect each other.

Therefore the three sides of the triangles AEB, BED, DEC, CEA are all equal.

Therefore the four angles at E are equal. Hence each of them is a right angle.

9. The square upon one side of a triangle is known to be equal to the sum of the squares on the two other sides; prove that the angle contained by those two sides must be a right angle. (See *Geom.*, Prop. 48.)

Section III.—Algebra.

10. If $a = 4$; $b = 3$; $c = 11$; $x = 6$; find the numerical value of

$$\frac{\sqrt{ab + 4x} - \sqrt{a^2 + b^2}}{\sqrt{a^2b^2 - 4c} - \sqrt{7x + 2(ab - 1)}}.$$

(See *Alg.*, § 1—13.)

11. Divide $x^5 + xy^4 - y^5$ by $x^2 - xy + y^2$.

12. Find the least common multiple of
$7(a^2 - x^2)$, $9(a + x)^2$, $15(a - x)$, $3(a^2 + x^2)$, $21(a^4 - x^4)$.

13. Solve two of the equations:—

(1.) $\frac{1}{2}(5x - 1) - 6(22 - 3x) = 2x - 3$;

(2.) $\frac{x}{x + a} - \frac{x - a}{x} = \frac{a^2}{(x - a)^2}$;

(3.) $\left\{\begin{array}{ll} \frac{x}{5} - \frac{y}{7} = 1 \ldots\ldots\ldots\ldots & \text{(i.)} \\ \frac{y}{5} - \frac{x}{3} = 2 \ldots\ldots\ldots\ldots & \text{(ii.)} \end{array}\right\}$

Multiply by the L.C.D's.

$$\left.\begin{array}{l} 7x - 5y = 35 \\ 3y - 5x = 30 \end{array}\right\}$$

Multiply the first by 5, and the second by 7;

$$\left.\begin{array}{r} 35x - 25y = 175 \\ -35x + 21y = 210 \end{array}\right\}$$

and so forth.

14. A train carrying three classes of passengers at 6d., 4d., and 3d., has eight times as many third-class passengers as there are of the second-class; and seven times as many second-class passengers as there are of the first-class. The whole sum received was £19. 7s. 2d. How many first-class tickets were issued?

Let x = the number of first-class passengers,
then $7x$ = the number of second-class passengers,
and $56x$ = the number of third-class passengers.
$\therefore$ $6x + 28x + 168x$ = the number of pence in £19. 7s. 2d. And so on.

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